(c) $\displaystyle\int_{-\infty}^{\infty} \frac{6x^5}{(x^6+3)^3}\,dx = \int_{-\infty}^{0} + \int_{0}^{\infty} = \lim_{t\to-\infty} \int_{t}^{0} \frac{5x^4}{(x^5+3)^3}\,dx + \lim_{s\to\infty} \int_{0}^{s} \frac{5x^4}{(x^5+3)^3}\,dx$

$$\left|\begin{array}{l} u = x^6+3 \\ du = 6x^5\,dx \\ x=t \Rightarrow u = t^6+3 \\ x=s \Rightarrow u = s^6+3 \\ x=0 \Rightarrow u=3 \end{array}\right|$$

$$= \lim_{t\to-\infty} \int_{t^6+3}^{3} \frac{du}{u^3} + \lim_{s\to\infty} \int_{3}^{s^6+3} \frac{du}{u^3}$$

$(p = 3 > 1)$ convergent.

$$= \lim_{t\to-\infty} \left.\frac{u^{-2}}{-2}\right|_{t^6+3}^{3} + \lim_{s\to\infty} \left.\frac{u^{-2}}{-2}\right|_{3}^{s^6+3}$$

$$= -\frac{1}{2(3^2)} + \frac{1}{2}\lim_{t\to-\infty} \cancel{\frac{1}{(t^6+3)^2}} \to 0 \;-\; \frac{1}{2}\lim_{s\to\infty} \cancel{\frac{1}{(s^6+3)^2}} \to 0 \;+\; \frac{1}{2(3^2)} = 0$$

(d) $\displaystyle\int_0^{2020} \frac{1}{\sqrt{2020-x}}\,dx$

discontinuity @ $x=2020$

$p=1/2<1$ convergent

$$= \lim_{t\to 2020^-} \int_0^t \frac{1}{\sqrt{2020-x}}\,dx$$

$u = 2020 - x$

$du = -dx$

$x=0 \Rightarrow u=2020$

$x=t \Rightarrow u=2020-t$

$$= -\lim_{t\to 2020^-} \int_{2020}^{2020-t} u^{-1/2}\,du$$

$$= -\lim_{t\to 2020^-} \frac{u^{1/2}}{1/2}\Bigg|_{2020}^{2020-t} = -2\left(\lim_{t\to 2020^-} \underbrace{\sqrt{2020-t}}_{0} - \sqrt{2020}\right) = \boxed{2\sqrt{2020}}$$

3. Find the following limits

(a) $\lim_{n\to\infty} \frac{(-1)^n}{n^3}$

Theorem If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

$\lim_{n\to\infty} \left| \frac{(-1)^n}{n^3} \right| = \lim_{n\to\infty} \frac{1}{n^3} = 0$, thus $\boxed{\lim_{n\to\infty} \frac{(-1)^n}{n^3} = 0}$

(b) $\lim_{n\to\infty} \frac{\sqrt{n}}{\ln n} = \frac{\infty}{\infty} \overset{L'H.R.}{=\!=} \lim_{n\to\infty} \frac{\frac{1}{2\sqrt{n}}}{\frac{1}{n}}$

$$= \lim_{n\to\infty} \frac{n}{2\sqrt{n}}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n}}{2}$$

$$= \infty$$

(c) $\lim_{n\to\infty} \frac{1-2n^2}{\sqrt[3]{n^6+1}+2n^2}$

$$= \lim_{n\to\infty} \frac{n^2\left(\frac{1}{n^2}-2\right)}{\sqrt[3]{n^6\left(1+\frac{1}{n^6}\right)}+2n^2}$$

$$= \lim_{n\to\infty} \frac{-2n^2}{\sqrt[3]{n^6}+2n^2}$$

$$= \lim_{n\to\infty} \frac{-2n^2}{n^2+2n^2}$$

$$= \lim_{n\to\infty} \frac{-2n^2}{3n^2}$$

$$= \boxed{-\frac{2}{3}}$$

$n\to\infty \quad \sqrt[3]{n^6} + 1 + 2n^2$

(d) $\displaystyle\lim_{n\to\infty}\left(\frac{1}{3}\ln(n^3+5n-2)-\ln(2-n)\right)$

$$= \lim_{n\to\infty}\left(\ln\sqrt[3]{n^3+5n-2} - \ln(2-n)\right)$$

$$= \lim_{n\to\infty} \ln\frac{\sqrt[3]{n^3+5n-2}}{2-n}$$

$$= \ln\left(\lim_{n\to\infty}\frac{\sqrt[3]{n^3+5n-2}}{2-n}\right)$$

$$= \ln\left(\lim_{n\to\infty}\frac{\sqrt[3]{n^3\left(1+\cancelto{0}{\frac{5}{n^2}}-\cancelto{0}{\frac{2}{n^3}}\right)}}{n\left(\cancelto{0}{\frac{2}{n}}-1\right)}\right) = \ln\left(\lim_{n\to\infty}\frac{\sqrt[3]{n^3}}{-n}\right)$$

$$= \ln\left(\lim_{n\to\infty}\frac{n}{-n}\right) = \ln(-1) \quad \boxed{D.N.E.}$$

4. Show that the sequence defined by $a_1 = 3$ and $a_{n+1} = 6 - \dfrac{8}{a_n}$ is increasing and bounded above. Find its limit.

$a_1 = 3$, $a_2 = 6 - \frac{8}{3} = \frac{10}{3} > 3 \Rightarrow a_1 < a_2$

Math. Induction: let $a_1 < a_2 < \dots < a_{n-1} < a_n$

show that $a_{n+1} > a_n$.

$a_{n+1} = 6 - \frac{8}{a_n}$, $a_n = 6 - \frac{8}{a_{n-1}}$

since $a_{n-1} < a_n$, then

$$\frac{8}{a_n} < \frac{8}{a_{n-1}}$$

and $\underbrace{6 - \frac{8}{a_n}}_{a_{n+1}} > \underbrace{6 - \frac{8}{a_n}}_{a_n}$

Bounded above: since a_n is increasing, then $\boxed{3 = a_1 < a_2 < \dots < a_n < \dots}$

for all n,

then $6 - \frac{8}{a_n} < 6$ (bounded above).

let $\lim_{n\to\infty} a_n = L$, then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(6 - \frac{8}{a_{n-1}}\right)$$

$$= 6 - \frac{8}{\lim_{n\to\infty} a_{n-1}} \quad \text{or}$$

$L = 6 - \frac{8}{L} \Rightarrow L^2 = 6L - 8$

$L^2 - 6L + 8 = 0$

$(L-2)(L-4) = 0$

$\cancel{L=2}$ not valid $\boxed{L=4}$

5. If the series $\sum_{n=1}^{\infty} a_n$ has a partial sum of $s_n = \frac{n}{2n+1}$, find a_4 and the sum of the series.

$S_n = a_n + S_{n-1}$, then $a_n = S_n - S_{n-1}$

$a_4 = S_4 - S_3 = \frac{4}{2(4)+1} - \frac{3}{2(3)+1}$

$= \frac{4}{9} - \frac{3}{7} = \frac{28-27}{63} = \boxed{\frac{1}{63}}$

Sum of the series $S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n}{2n+1} = \boxed{\frac{1}{2}}$

6. Find the sum of the series.

(a) $\sum_{n=1}^{\infty} \frac{2^{2n+1}}{3^{3n-1}}$

$$= \sum_{n=1}^{\infty} \frac{2 \cdot 2^{2n}}{\frac{1}{3} 3^{3n}}$$

$$= \sum_{n=1}^{\infty} 6 \left(\frac{2^2}{3^3}\right)^n$$

$$= \sum_{n=1}^{\infty} 6 \left(\frac{4}{27}\right)^n$$

$$= \sum_{n=1}^{\infty} 6 \left(\frac{4}{27}\right) \left(\frac{4}{27}\right)^{n-1}$$

$$= \frac{6 \frac{4}{27}}{1 - \frac{4}{27}}$$

$$= \frac{\frac{24}{27}}{\frac{23}{27}} = \boxed{\frac{24}{23}}$$

(b) $\sum_{n=3}^{\infty} \frac{1}{n^2-4}$ partial fractions:

$$\frac{1}{n^2-4} = \frac{1}{(n-2)(n+2)} = \frac{A}{n-2} + \frac{B}{n+2}$$

$$= \frac{A(n+2)+B(n-2)}{(n-2)(n+2)}$$

$$1 = A(n+2) + B(n-2)$$

$n=-2$: $1 = -4B \longrightarrow B = -\frac{1}{4}$

$n=2$: $1 = 4A \longrightarrow A = \frac{1}{4}$

$$\frac{1}{n^2-4} = \frac{1}{4}\left(\frac{1}{n-2} - \frac{1}{n+2}\right)$$

Partial sums:

$$S_3 = \frac{1}{4}\left(\frac{1}{1} - \frac{1}{5}\right) = a_3$$

$$a_3 + a_4 = S_4 = \frac{1}{4}\left(\frac{1}{1} - \frac{1}{5}\right) + \frac{1}{4}\left(\frac{1}{2} - \frac{1}{6}\right)$$

$$= \frac{1}{4}\left(\frac{1}{1} + \frac{1}{2} - \frac{1}{5} - \frac{1}{6}\right)$$

$$S_5 = a_3 + a_4 + a_5 = \frac{1}{4}\left(\frac{1}{1} + \frac{1}{2} - \frac{1}{5} - \frac{1}{6}\right) + \frac{1}{4}\left(\frac{1}{3} - \frac{1}{7}\right)$$

$$S_6 = a_3 + a_4 + a_5 + a_6 = \frac{1}{4}\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} - \frac{1}{5} - \frac{1}{6} - \frac{1}{7}\right)$$

$$+ \frac{1}{4}\left(\frac{1}{4} - \frac{1}{8}\right)$$

$$S_7 = a_3 + a_4 + a_5 + a_6 + a_7 = \frac{1}{4}\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \cancel{\frac{1}{5}} - \frac{1}{6} - \frac{1}{7} - \frac{1}{8}\right)$$

$$+ \frac{1}{4}\left(\cancel{\frac{1}{8}} - \frac{1}{9}\right)$$

$$= \frac{1}{4}\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{6} - \frac{1}{7} - \frac{1}{8} - \frac{1}{9}\right)$$

$$S_n = \frac{1}{4}\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{n-1} - \frac{1}{n} - \frac{1}{n+1} - \frac{1}{n+2}\right)$$

$$S = \lim_{n\to\infty} S_n = \frac{1}{4} \lim_{n\to\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{n-1} - \frac{1}{n} - \frac{1}{n+1} - \frac{1}{n+2}\right)$$

$$= \frac{1}{4}\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right)$$

$$= \frac{1}{4}\,\frac{24+12+8+6}{24}$$

$$= \frac{1}{4}\,\frac{50}{24}$$

$$= \boxed{\frac{25}{48}}$$

19. Find the sum of the series

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{2^{2n+1}}{3^{3n-1}}$

(b) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n^2-4}$

see practice problems E.3.

(c) $\displaystyle\sum_{n=2}^{\infty} \frac{3^n}{5^n n!} = \sum_{n=2}^{\infty} \left(\frac{3}{5}\right)^n \cdot \frac{1}{n!} = \boxed{e^{3/5} - 1 - \frac{3}{5}}$

$\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$ $x = \frac{3}{5}$

$\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \sum_{n=2}^{\infty} \frac{x^n}{n!} \Rightarrow \sum_{n=2}^{\infty} \frac{x^n}{n!} = e^x - 1 - x$

7. Which of the following series is convergent?

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^{5/7}+1}$

$$\lim_{n\to\infty} \frac{n^2}{n^{5/7}+1} = \lim_{n\to\infty} \frac{n^2}{n^{5/7}\left(1 - \frac{1}{n^{5/7}}\right)}$$

$$= \lim_{n\to\infty} \frac{n^2}{n^{5/7}} = \lim_{n\to\infty} n^{9/7} = \infty$$

$\boxed{\text{Divergent by Divergence Test}}$

(b) $\sum_{n=1}^{\infty} \frac{\cos^2 n}{3^n}$

$0 \le \cos^2 n \le 1$

$\frac{\cos^2 n}{3^n} \le \frac{1}{3^n}$

compare with $\sum_{n=1}^{\infty} \frac{1}{3^n}$ (geometric, $r = 1/3$, convergent)

$\boxed{\text{Convergent by Comparison Test}}$

(c) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

$f(x) = \frac{1}{x(\ln x)^2}$ is positive, continuous and decreasing on $[2, \infty)$.

$$\int_2^{\infty} \frac{dx}{x(\ln x)^2} = \lim_{t \to \infty} \int_2^{t} \frac{dx}{x(\ln x)^2}$$

$u = \ln x$
$du = \frac{dx}{x}$
$x = 2 \to u = \ln 2$
$x = t \to u = \ln t$

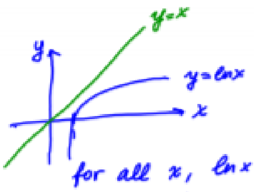
$$= \lim_{t \to \infty} \int_{\ln 2}^{\ln t} \frac{du}{u^2} \quad \left(\begin{array}{l} p = 2 > 1 \\ \text{so convergent} \end{array} \right)$$

$$\boxed{\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2} \text{ convergent by Integral Test}}$$

8. Which of the following series is absolutely convergent

(a) $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$

$a_n = \frac{(-1)^n}{\ln n}$



$\sum_{n=2}^{\infty} \left| \frac{(-1)^n}{\ln n} \right| = \sum_{n=2}^{\infty} \frac{1}{\ln n}$

for all x, $\ln x < x$

$\frac{1}{\ln x} > \frac{1}{x}$

Comparison Test.

$\frac{1}{n} < \frac{1}{\ln n}$

$\sum_{n=2}^{\infty} \frac{1}{n}$ - divergent (harmonic series)

By Comparison Test, $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ is divergent.

$\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ - alternating series, $b_n = \frac{1}{\ln n}$

- $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\ln n} = 0$
- $\left(\frac{1}{\ln n}\right)' = -\frac{\frac{1}{n}}{(\ln n)^2} = -\frac{1}{n(\ln n)^2} < 0$ on $[2, \infty)$

b_n is decreasing

$\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ is convergent by the alternating series Test.

convergent but not absolutely convergent = $\boxed{\text{conditionally convergent}}$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-3)^n}{n!}$

Ratio Test. $a_n = \frac{(-3)^n}{n!}$, $a_{n+1} = \frac{(-3)^{n+1}}{(n+1)!}$

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\frac{(-3)^{n+1}}{(n+1)!}}{\frac{(-3)^n}{n!}} \right| = \lim_{n\to\infty} \left| \frac{(-3)^{\cancel{n+1}}}{(n+1)!} \frac{n!}{\cancel{(-3)^n}} \right| \xrightarrow{(n+1)! = (n+1)n!} \lim_{n\to\infty} \left| \frac{-3}{n+1} \right|$

$= 0 < 1$

$\boxed{\text{absolutely convergent by Ratio Test}}$

(c) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$

$a_n = (-1)^{n-1}\frac{1}{n}$

$\sum_{n=1}^{\infty}\left|(-1)^{n-1}\frac{1}{n}\right| = \sum_{n=1}^{\infty}\frac{1}{n}$ divergent (harmonic, $p=1$)

$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ - alternating series, $b_n = \frac{1}{n}$

- $\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{1}{n} = 0$
- $\left(\frac{1}{n}\right)' = -\frac{1}{n^2} < 0$ on $[1,\infty)$

$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ is convergent by Alternating Series Test.

$\boxed{\text{conditionally convergent}}$

(d) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{\sqrt{n-2}}$.

$$\sum_{n=1}^{\infty}\left|(-1)^{n-1}\frac{n}{\sqrt{n-2}}\right| = \sum_{n=1}^{\infty}\frac{n}{\sqrt{n-2}}$$

$$\lim_{n\to\infty}\frac{n}{\sqrt{n-2}} = \infty$$

Divergent by Divergence Test.

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{\sqrt{n-2}} \quad \text{alternating,} \quad b_n = \frac{n}{\sqrt{n-2}}$$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{n}{\sqrt{n-2}} = \infty$$

$\boxed{\text{Divergent}}$

(e) $\sum_{n=0}^{\infty}(-1)^n \frac{2^{2n}}{3^{3n}} = \sum_{n=0}^{\infty} (-1)^n \left(\frac{4}{27}\right)^n$ — $\boxed{\text{absolutely convergent}}$

Ratio Test, or do $\sum_{n=0}^{\infty}\left(\frac{4}{27}\right)^n$ - geometric, $r = \frac{4}{27} < 1$

convergent.

9. Find the radius of convergence and interval of convergence of the series $\sum_{n=1}^{\infty} \frac{2^n(x-3)^n}{\sqrt{n+3}}$.

The radius of converges

$$R = \lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right|, \text{ where } c_n = \frac{2^n}{\sqrt{n+3}}$$

$$R = \lim_{n\to\infty} \left| \frac{2^n}{\sqrt{n+3}} \cdot \frac{\sqrt{n+4}}{2^{n+1}} \right| = \frac{1}{2}.$$

The interval of convergence:

$$|x-3| < \frac{1}{2}$$

$$-\frac{1}{2} < x-3 < \frac{1}{2}$$

$$+\frac{5}{2} < x < \frac{7}{2}$$

End points: $x = +\frac{5}{2} \to \sum_{n=1}^{\infty} \frac{2^n\left(+\frac{5}{2}-3\right)^n}{\sqrt{n+3}}$

$$= \sum_{n=1}^{\infty} \cancel{(-1)^n} \frac{2^n\left(-\frac{1}{2}\right)^n}{\sqrt{n+3}}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+3}}$$ – converges but not absolutely.

$x = \frac{7}{2}$: $\sum_{n=1}^{\infty} \frac{2^n\left(\frac{7}{2}-3\right)^n}{\sqrt{n+3}} = \sum_{n=1}^{\infty} \frac{2^n\left(\frac{1}{2}\right)^n}{\sqrt{n+3}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+3}}$ diverges.

interval of convergence: $\boxed{\left[\frac{5}{2}, \frac{7}{2}\right)}$.

$\boxed{R = \frac{1}{2}}$

4

$n=1$

10. Find the Maclaurin series for the function

(a) $f(x) = \ln(3-2x)$

(b) $f(x) = \dfrac{x^2}{(1+9x)^3}$

$$\frac{1}{1-x} = \sum_{n=1} x^n \qquad |x|<1$$

$$f(x) = \frac{2x}{4+x^3} = 2x \cdot \frac{1}{4\left(1+\frac{x^3}{4}\right)} \qquad \left|\frac{x^3}{4}\right|<1$$

$$= \frac{2x}{4} \cdot \frac{1}{1-\left(-\frac{x^3}{4}\right)} = \frac{x}{2} \cdot \sum_{n=1}^{\infty} \left(-\frac{x^3}{4}\right)^n$$

$$= \frac{x}{2} \sum_{n=1}^{\infty} (-1)^n \frac{(x^3)^n}{4^n}$$

$$= \frac{x}{2} \sum_{n=1}^{\infty} (-1)^n \frac{x^{3n}}{4^n}$$

$$= \boxed{\frac{1}{2} \sum_{n=1}^{\infty} (-1)^n \frac{x^{3n+1}}{4^n}} \checkmark$$

$$= \sum_{n=1}^{\infty} (-1)^n \frac{x^{3n+1}}{4^n \cdot 2} \qquad \left(\begin{array}{l}4=2^2\\4^n=2^{2n}\end{array}\right)$$

$$= \boxed{\sum_{n=1}^{\infty} (-1)^n \frac{x^{3n+1}}{2^{2n+1}}}$$

$\ln(3-2x)$

$$\boxed{\int \frac{du}{1+u} = \ln u + C}$$

$1+u = 3-2x$

$u = 2-2x$

$du = -2dx$

$$\int \frac{-2dx}{3-2x} = \ln(3-2x)+C$$

$$\ln(3-2x) = C + \int \frac{-2dx}{3-2x}$$

$$-\frac{2}{3-2x} = -2 \cdot \frac{1}{3\left(1-\frac{2}{3}x\right)}$$

$$= -\frac{2}{3}\frac{1}{1-\frac{2}{3}x} = -\frac{2}{3}\sum_{n=1}^{\infty}\left(\frac{2}{3}x\right)^n$$

$$= -\frac{2}{3}\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n x^n$$

$$= -\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^{n+1} x^n$$

$\left|\frac{2}{3}x\right| < 1$

$R = \frac{3}{2}$

$$\ln(3-2x) = C - \sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^{n+1}\left(\int x^n dx\right)$$

$$\ln(3-2x) = C - \sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^{n+1}\frac{x^{n+1}}{n+1}$$

@ $x=0$

$\ln 3 = C$

$$\boxed{\ln(3-2x) = (\ln 3) - \sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^{n+1}\frac{x^{n+1}}{n+1}}$$

23. Find a power series centered at $x = 0$ for the given function function and determine the radius of convergence.

(a) $f(x) = \dfrac{x}{1 - 8x^3}$ $= x \sum_{n=0}^{\infty} (8x^3)^n = x \sum_{n=0}^{\infty} 8^n x^{3n} = \sum_{n=0}^{\infty} 8^n x^{3n+1}$

(b) $f(x) = \ln(3 - 2x)$

(c) $f(x) = \dfrac{x^2}{(1 + 9x)^3}$

see practice problems E.3.

$(1+9x)^3$

24. Find the Taylor series for the function $f(x) = \sqrt{x}$ at $a = 16$.

$f(x) = \sqrt{x}, \; a = 16$

$f'(x) = \frac{1}{2} x^{-1/2}$

$f''(x) = \frac{1}{2}\left(-\frac{1}{2}\right) x^{-1/2 - 1}$

$f'''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right) x^{-5/2} = \frac{3(-1)^2}{2^3} x^{-5/2}, \; n = 3.$

$f^{iv}(x) = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right) x^{-7/2}.$

$f^{(n)} = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right) \cdots \left(-\frac{2n-3}{2}\right) x^{-\frac{(2n-1)}{2}}$

$n = 3 \longrightarrow 3$

$n = 4 \longrightarrow 5$

$2n+1$

$2n-1$

$2n-3$

$f^{(n)}(16) = \frac{(-1)^{n-1} (1)(3)(5)\cdots(2n+3)}{2^n} \frac{1}{4^{2n+1}}$

$= \frac{(-1)^{n-1} (1)(3)(5)\cdots(2n+3)}{2^n} \frac{1}{(2^2)^{2n+1}}$

$= \frac{(-1)^{n-1} (1)(3)(5)\cdots(2n+3)}{2^{4n+2+n}}$

$= \frac{(-1)^{n-1} (1)(3)(5)\cdots(2n+3)}{2^{5n+2}}$

Double factorials: $(2n)!! = (2)(4)(6)(8)\ldots(2n-2)(2n)$

$(2n+1)!! = (1)(3)(5)\cdots(2n-1)(2n+1)$

$$\boxed{f^{(n)}(16) = \frac{(-1)^{n-1} (2n+3)!!}{2^{5n+2}}}.$$

$$\sqrt{x} = \sum_{n=0}^{\infty} \frac{f^{(n)}(16)}{n!} (x-16)^n = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^{n-1} (2n+3)!!}{2^{5n+2} \cdot n!} (x-16)^n}$$

$$\int \frac{\cos x - 1}{x}\,dx$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \dots$$

$$\cos x - 1 = \left(\cancel{1} - \frac{x^2}{2} + \frac{x^4}{4!} - \dots\right) - \cancel{1}$$

$$= -\frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$\frac{\cos x - 1}{x} = -\frac{x}{2} + \frac{x^3}{4!} - \frac{x^5}{6!} + \dots = \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n-1}}{(2n)!}$$

$$\int_0^{1/3} \frac{1}{1+x^7}\,dx$$

$$\frac{1}{1+x^7} = \sum_{n=0}^{\infty} (-1)^n x^{7n}$$

26. Evaluate the integral $\int_0^{1/3} \frac{1}{1+x^7} dx$ as an infinite series.

$$\frac{1}{1+x^7} = \sum_{n=0}^{\infty} (-1)^n x^{7n} \quad (\text{see } 11.9)$$

$$\int_0^{1/3} \frac{1}{1+x^7} dx = \int_0^{1/3} \left(\sum_{n=0}^{\infty} (-1)^n x^{7n} \right) dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \left(\int_0^{1/3} x^{7n} dx \right)$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{7n+1}}{7n+1} \Big|_0^{1/3}$$

$$= \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{1}{7n+1} \left(\frac{1}{3} \right)^{7n+1}}$$

$f(x) = \sin x, \quad a = \pi.$

$f(x) = \sin x \quad = f^{iv}$

$f'(x) = \cos x \quad = f^{v}$

$f''(x) = -\sin x \quad = \ldots$

$f'''(x) = -\cos x \quad \ldots$

$f(\pi) = 0 \Rightarrow$ even n

$f'(\pi) = -1 \leftarrow \quad n = 1, 5, 9, \ldots$

$f''(\pi) = 0 \Rightarrow$ even n

$f'''(\pi) = 1 \leftarrow \quad n = 3, 7, 11, \ldots$

only odd n,

$n = 2k + 1$

$f^{(2k+1)}(x) = (-1)^{k+1}$

$$\boxed{\sin x = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(2k+1)!} (x - \pi)^{2k+1}}$$

25. Find the Maclaurin series for the function $f(x) = x^2 \ln(1 + x^3)$.

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n}$$

$$\ln(1+x^3) = \sum_{n=1}^{\infty} (-1)^n \frac{(x^3)^n}{n} = \sum_{n=1}^{\infty} (-1)^n \frac{x^{3n}}{n}$$

$$x\ln(1+x^3) = x\sum_{n=1}^{\infty} (-1)^n \frac{x^{3n}}{n}$$

$$= \boxed{\sum_{n=1}^{\infty} (-1)^n \frac{x^{3n+1}}{n}}$$

$$\frac{x^2}{(1+9x)^3}$$

$$\left(\frac{1}{1+9x}\right)' = -\frac{9}{(1+9x)^2}$$

$$\left(-\frac{9}{(1+9x)^2}\right)' = -9\left((1+9x)^{-2}\right)'$$

$$= -9(-2)(1+9x)^{-3}(9)$$

$$\left(\frac{1}{1+9x}\right)'' = \frac{162}{(1+9x)^3} \Rightarrow \frac{1}{(1+9x)^3} = \frac{1}{162}\left(\frac{1}{1+9x}\right)''$$

$$\boxed{\frac{x^2}{(1+9x)^3} = \frac{x^2}{162}\left(\frac{1}{1+9x}\right)''}$$

$$\frac{1}{1+9x} = \frac{1}{1-(-9x)} = \sum_{n=1}^{\infty}(-9x)^n = \sum_{n=1}^{\infty}(-1)^n 9^n x^n$$

$$\boxed{\begin{array}{l} |9x|<1 \\ |x|<\frac{1}{9} \\ R=\frac{1}{9} \end{array}}$$

$$\left(\frac{1}{1+9x}\right)' = \left(\sum_{n=1}^{\infty}(-1)^n 9^n x^n\right)'$$

$$= \sum_{n=1}^{\infty}(-1)^n 9^n n x^{n-1}$$

$$\left(\frac{1}{1+9x}\right)'' = \left(\sum_{n=1}^{\infty}(-1)^n 9^n n (x^{n-1})\right)'$$

$$= \sum_{n=2}^{\infty}(-1)^n 9^n n(n-1) x^{n-2}$$

$$\frac{x^2}{(1+9x)^3} = \frac{x^2}{162}\sum_{n=2}^{\infty}(-1)^n 9^n n(n-1) x^{n-2}$$

$$= \boxed{\frac{1}{162}\sum_{n=2}^{\infty}(-1)^n 9^n n(n-1) x^n}$$

7. Find the length of the curve

(a) $x(t) = 3t - t^3, \quad y(t) = 3t^2, \quad 0 \le t \le 2$ | $x'(t) = 3 - 3t^2, \quad y' = 6t$

$$L = \int_0^2 \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt = \int_0^2 \sqrt{(3-3t^2)^2 + (6t)^2}\, dt$$

$$= \int_0^2 \sqrt{9 - 18t^2 + 9t^4 + 36t^2}\, dt = \int_0^2 \sqrt{9 + 18t^2 + 9t^4}\, dt = \int_0^2 \sqrt{(3+3t^2)^2}\, dt$$

$$= \int_0^2 (3 + 3t^2)\, dt = \left(3t + \frac{3t^3}{3}\right)\Big|_0^2 = 6 + 8 = \boxed{14}$$

(b) $y = \frac{1}{4}x^2 - \frac{1}{2}\ln x, \quad 1 \le x \le 2, \quad y' = \frac{2x}{4} - \frac{1}{2x} = \frac{1}{2}\left(x - \frac{1}{x}\right)$

$$L = \int_1^2 \sqrt{1 + [y'(x)]^2}\, dx = \int_1^2 \sqrt{1 + \frac{1}{4}\left(x - \frac{1}{x}\right)^2}\, dx$$

$$= \int_1^2 \sqrt{1 + \frac{1}{4}\left(x^2 - 2 + \frac{1}{x^2}\right)}\, dx = \int_1^2 \sqrt{1 + \frac{x^2}{4} - \frac{1}{2} + \frac{1}{4x^2}}\, dx$$

$$= \int_1^2 \sqrt{\frac{x^2}{4} + \frac{1}{2} + \frac{1}{4x^2}}\, dx = \int_1^2 \sqrt{\frac{1}{4}\left(x + \frac{1}{x}\right)^2}\, dx = \frac{1}{2}\int_1^2 \left(x + \frac{1}{x}\right) dx$$

$$= \frac{1}{2}\left[\frac{x^2}{2} + \ln x\right]_1^2 = \boxed{\frac{1}{2}\left(2 - \frac{1}{2} + \ln 2\right)}$$

1. Find the area of the surface obtained by rotating the curve $y = x^3$, $0 \leq x \leq 2$ about the x-axis.

$$S_x = 2\pi \int_0^2 y(x)\sqrt{1 + [y'(x)]^2}\,dx$$

$$y'(x) = 3x^2$$

$$= 2\pi \int_0^2 x^3 \sqrt{1+9x^4}\,dx$$

$u = 1+9x^4$
$du = 36x^3\,dx$
$x=0 \longrightarrow u=1$
$x=2 \longrightarrow u = 1+9(16)$
$= 145$

$$= \frac{2\pi}{36}\int_1^{145} \sqrt{u}\,du$$

$$= \frac{\pi}{18}\,\frac{2u^{3/2}}{3}\Bigg|_1^{145}$$

$$= \boxed{\frac{\pi}{27}\left((145)^{3/2} - 1\right)}$$

2. Find the area of the surface obtained by rotating the curve $x = \sqrt{2y - y^2}$, $0 \le y \le 1$ about the y-axis.

$$S_y = 2\pi \int_0^1 x(y)\sqrt{1 + (x'(y))^2}\, dy$$

$$x'(y) = \frac{1}{2\sqrt{2y-y^2}}(2-2y) = \frac{1-y}{\sqrt{2y-y^2}}$$

$$= 2\pi \int_0^1 \sqrt{2y-y^2}\sqrt{1 + \frac{(1-y)^2}{2y-y^2}}\, dy$$

$$= 2\pi \int_0^1 \sqrt{2y-y^2}\sqrt{\frac{2y-y^2+1-2y+y^2}{2y-y^2}}\, dy$$

$$= 2\pi \int_0^1 \sqrt{2y-y^2}\sqrt{\frac{1}{2y-y^2}}\, dy$$

$$= \boxed{2\pi}$$