Math 308: Week-in-Review 9 (6.6, 5.1, 5.2)

# 1 6.6: Convolution

## Review

- The convolution of two functions $f(t)$ and $g(t)$, denoted $(f * g)(t)$, is defined by

$$(f * g)(t) = \int_0^t f(x)g(t-x)\,dx$$

  for $t \geq 0$, assuming both functions are zero for $t < 0$.

- The Laplace transform of a convolution $(f * g)(t)$ is

$$\mathcal{L}\{(f * g)(t)\} = F(s) \cdot G(s),$$

  where $F(s) = \mathcal{L}\{f(t)\}$ and $G(s) = \mathcal{L}\{g(t)\}$.

- This property simplifies solving differential equations by converting convolution in the time domain to multiplication in the s-domain.

- Convolution is commutative: $f * g = g * f$ so

$$(f * g)(t) = \int_0^t f(x)g(t-x)\,dx = \int_0^t f(t-x)g(x)\,dx.$$

1. Find the following convolutions using the definition only

   (a) $e^{2t} * e^{4t}$

   (b) $t^2 * t$,

2. Using the Laplace transform (instead of the definition) compute the following convolutions

   (a) $u_2(t) * u_3(t)$

   (b) $t^2 * t$,

3. In each of the following cases find a function (or generalized function) $g(t)$ that satisfies the equality for $t \geq 0$

   (a) $t^2 * g(t) = t^5$

   (b) $1 * 1 * g(t) = t^3$

   (c) $1 * g(t) = t$

4. Write the inverse Laplace transform in terms of a convolution integral

$$F(s) = \frac{s^2}{(s+2)^3(s+5)^2}$$

5. Solve the initial value problem

$$y'' - 3y' + 2y = h(t), \;\; y(0) = 2, \;\; y'(0) = -1.$$

## 2 5.1–5.2: Power Series Solutions of Linear Differential Equations

### Review

- A power series solution of a linear differential equation assumes the solution can be written as

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

where $x_0$ is the center of the series (often $x_0 = 0$), and $a_n$ are coefficients to be determined.

- The derivatives of the power series are:

$$y'(x) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1}, \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n (x - x_0)^{n-2},$$

obtained by term-by-term differentiation, assuming the series converges in some interval.

- For a linear differential equation of the form $P(x)y'' + Q(x)y' + R(x)y = 0$, substitute the power series for $y$, $y'$, and $y''$ into the equation, equate coefficients of like powers of $(x - x_0)$, and solve for the recurrence relation among the $a_n$.

- A point $x_0$ is an *ordinary point* if $P(x_0) \neq 0$ and the coefficients $Q(x)/P(x)$ and $R(x)/P(x)$ are analytic at $x_0$. In this case, the series solution converges in some interval around $x_0$.

- The radius of convergence of the series solution is at least as large as the distance from $x_0$ to the nearest singular point (where $P(x) = 0$), determined by analyzing the coefficient functions.

- Solutions typically yield two linearly independent series $y_1(x)$ and $y_2(x)$, whose Wronskian $W[y_1, y_2](x_0) \neq 0$ confirms their independence.

6. Determine the radius of convergence for the power series

   (a) $\displaystyle\sum_{n=0}^{\infty} \frac{x^{3n}}{n!}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n^3 (x-1)^n}{4^n}$

7. For the equation $y'' - xy' + xy = 0$

   (a) Determine a lower bound for the radius of convergence for the series solutions about $x_0 = 0$.

   (b) Seek its power series solution about $x_0 = 0$. Find the recurrence relation.

   (c) Find the general term of each solution $y_1(x)$ and $y_2(x)$.

   (d) Find the first four terms in each of the solutions. Show that $W[y_1, y_2](0) \neq 0$.