Math 308: Week-in-Review 6 (EXAM 1 REVIEW)

Characterizing Differential Equations

Review

- The order of a differential equation is the order of the highest derivative.
- Ordinary vs Partial Differential Equations:
  - An ordinary differential equation has derivatives with respect to one variable.
  - A partial differential equation has derivatives with respect to more than one variable.
- Linear ODEs:
  - A linear ODE has the form

$$a_n(x)y^{(n)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) = g(x).$$

  - Conditions:
    - All the y's are in different terms.
    - None of the y's are inside a function or to a power.
    - The y's can be multiplied by a function of x.
    - There can be terms that depend only on x.
- Homogeneous Linear ODEs:
  - A linear ODE is homogeneous if the $g(x)$ term is 0.
- Separable ODEs:
  - An ODE is separable if you can write it in the form $y' = f(x)g(y)$.
- Autonomous ODEs:
  - An ODE is autonomous if the independent variable (x) does not show up explicitly, i.e., if x does not show up outside of y.

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1. Classify the following differential equations into one (or more) of the following categories and state the order: Partial differential equation, Ordinary differential equation, Separable, Linear, Homogeneous, Autonomous.

   (a) $y^2 - y'' + 6 = 0$

   - **Order**: 2
   - **Type**: Nonlinear ODE (due to y^2), autonomous, separable, non-homogeneous (due to 6)

   (b) $f_x - f_y = xf$

   - **Order**: 1
   - **Type**: Linear PDE, homogeneous, non-autonomous (due to xf)

   (c) $y'(x) + x^2y(x) = 3y(x)$

   - **Order**: 1
   - **Type**: Linear ODE, homogeneous, non-autonomous, separable since $y' = y(3-x^2)$

   (d) $g' = x^2 \sin(g)$

   - **Order**: 1
   - **Type**: Nonlinear ODE due to $\sin(g)$, separable, homogeneous, non-autonomous due to $x^2 \sin(g)$

   (e) $\sin(x)w''' + w - 3 = 0$

   - **Order**: 3
   - **Type**: Linear ODE, nonhomogeneous, separable since $w''' = \dfrac{3-w}{\sin(x)}$, non-autonomous due to $\sin(x)w'''$

   (f) $u''(x) = \sin(u(x))$

   - **Order**: 2
   - **Type**: Nonlinear ODE due to $\sin(u)$, autonomous, homogeneous, separable

   (g) $f^{(5)} - \cos(x^2)f''' - \tan(x)f = 3\tan(x)$

   - **Order**: 5
   - **Type**: Linear ODE, nonhomogeneous, non-autonomous

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Solving Differential Equations

Review

- First Order ODEs:
  - You do NOT need to guess which method to use to solve a 1st order ODE!
  - How to determine which method to use:
    (a) Is the equation separable? If yes, use separation of variables.
    (b) Is the equation linear? If yes, use the method of integrating factors.
    (c) Is it a Bernoulli equation? If yes, then use $v = y^{1-n}$.
    (d) Is the equation exact? If yes, then use the method for exact equations.
    (e) Is it a homogeneous equation? If yes, then use $v = y/x$ to get a separable equation.
    (f) If none of the above, then try to find an integrating factor to make the equation exact.
- Second Order Linear ODEs:
  - Homogeneous with constant coefficients:
    (a) Look for solutions of the form $y(t) = e^{rt}$.
    (b) Find the characteristic equation.
    (c) Find the roots of the characteristic equation.
    (d) The general solution is given by:
      * Distinct real roots: $c_1e^{r_1t} + c_2e^{r_2t}$
      * Complex roots: $c_1e^{at}\cos(bt) + c_2e^{at}\sin(bt)$
      * Repeated real roots: $c_1e^{rt} + c_2te^{rt}$
    (e) If you have initial conditions, use them to solve for c_1 and c_2.
  - Nonhomogeneous:
    * Method of undetermined coefficients (if constant coefficients and you can guess).
    * Variation of parameters.

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2. Find the general solution to
$$t^2y' + ty - t = 0.$$

Solution: Rewrite in standard form as:
$$y' + \frac{1}{t}y = \frac{1}{t}$$

Integrating factor: $\mu(t) = e^{\int \frac{1}{t}\,dt} = t$. Multiply through by $\mu(t) = t$:
$$ty' + y = 1 \implies \frac{d}{dt}(ty) = 1$$

Integrate:
$$ty = t + C \implies y = 1 + \frac{C}{t}$$

General solution: $\boxed{y(t) = 1 + \frac{C}{t}}$

3. Solve the initial value problem
$$u' - tu^{-2} = 0, \quad u(1) = -1.$$

Solution: Separate variables:
$$u^2\,du = t\,dt \implies \frac{u^3}{3} = \frac{t^2}{2} + C$$

Apply $u(1) = -1$:
$$\frac{(-1)^3}{3} = \frac{1}{2} + C \implies C = -\frac{5}{6}$$

Final Solution:
$$\boxed{u(t) = \left(\frac{3t^2}{2} - \frac{5}{2}\right)^{1/3} = \sqrt[3]{\frac{3t^2}{2} - \frac{5}{2}}}$$

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4. Find the general solution to
$$f'' = 3f' - 2f.$$

Solution: Rewrite
$$f'' - 3f' + 2f = 0$$
Characteristic equation:
$$\lambda^2 - 3\lambda + 2 = 0 \implies (\lambda - 1)(\lambda - 2) = 0 \implies \lambda = 1,\, 2$$
General solution:
$$\boxed{f(t) = C_1 e^t + C_2 e^{2t}}$$

5. Find the general solution to
$$w'' + 4w' + 4w = 5e^t.$$

Solution: Homogeneous solution:
$$\lambda^2 + 4\lambda + 4 = 0 \implies (\lambda + 2)^2 = 0 \implies \lambda = -2 \text{ (repeated root)}$$
$$w_c(t) = C_1 e^{-2t} + C_2 t e^{-2t}$$
Particular solution $w_p = Ae^t$ (since the right hand side is not a homogeneous solution). Taking derivatives,
$$w_p = w_p' = w_p'' = Ae^t$$
Plugging into the differential equation,
$$Ae^t(1 + 4 + 4) = 9Ae^t = 5e^t \implies A = \frac{5}{9}$$
General solution:
$$\boxed{w(t) = C_1 e^{-2t} + C_2 t e^{-2t} + \frac{5}{9} e^t}$$

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6. Find the general solution to
$$(4x-2y)y'+4y=-2x.$$

Solution: Rewrite as:
$$(2x+4y)+(4x-2y)y'=0$$
Set $M=2x+4y$ and $N=4x-2y$.

Check for exactness:
$$\frac{\partial M}{\partial y}=4 \text{ and } \frac{\partial N}{\partial x}=4$$
So the equation is exact. That means there exists a function $F(x,y)$ such that
$$\frac{\partial F}{\partial x}=M=2x+4y$$
and
$$\frac{\partial F}{\partial y}=N=4x-2y$$
Integrate $\frac{\partial F}{\partial x}=M=2x+4y$ with respect to the variable x:
$$F(x,y)=x^2+4xy+h(y) \implies \frac{\partial F}{\partial y}=4x+h'(y)=N=4x-2y$$
Hence
$$h'(y)=-2y \implies h(y)=-y^2+c$$
Solution:
$$\boxed{x^2+4xy-y^2=C}$$

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7. Find the general solution to
$$3g'' - 2g' + 4 = 0.$$

Solution: Rewrite as
$$3g'' - 2g' = -4$$

This a non-homogeneous linear equation. Solution is of the form

$$g(t) = g_c(t) + g_p(t)$$

where $g_c(t)$ is the solution of $3g'' - 2g' = 0$ and $g_p(t)$ is a particular solution.

To find $g_c(t)$: Characteristic equation:

$$3\lambda^2 - 2\lambda = 0 \implies \lambda = 0, \frac{2}{3}$$

So the general solution of the homogeneous equation is

$$g_c(t) = C_1 + C_2 e^{2t/3}$$

Since the right hand side is a constant, and one of the homogeneous functions is also a constant, multiply by t to get
$$g_p(t) = At$$

Then plugging into the differential equation, we get

$$g_p' = A \text{ and } g_p'' = 0$$

So
$$3g_p'' - 2g_p' = 3 \cdot 0 - 2A = -4 \implies A = 2$$

General solution:
$$\boxed{g(t) = 2t + C_1 + C_2 e^{2t/3}}$$

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8. Solve the initial value problem

$$f = -\frac{1}{9}f'', \quad f(0) = -2, \quad f'(0) = 1.$$

Solution: Rewrite as

$$f'' + 9f = 0, \quad f(0) = -2, \quad f'(0) = 1$$

Characteristic equation:

$$\lambda^2 + 9 = 0 \implies \lambda = \pm 3i$$

General Solution:

$$f(t) = C_1 \cos(3t) + C_2 \sin(3t)$$

Find the coefficients C_1, C_2. Applying initial conditions

$$f(0) = C_1 = -2$$

and

$$f'(t) = -3C_1 \sin(3t) + 3C_2 \cos(3t) \implies f'(0) = 3C_2 = 1 \implies C_2 = \frac{1}{3}$$

Final Solution:

$$\boxed{f(t) = -2\cos(3t) + \frac{1}{3}\sin(3t)}$$

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9. Suppose you wanted to use the method of undetermined coefficients to find a particular solution to

$$y'' - 5y' + 6y = 4e^{-2t} + 3t^3.$$

What is an appropriate guess for the particular solution y_p?

Solution:

(a) **Homogeneous Solution:**

$$y'' - 5y' + 6y = 0 \implies \lambda^2 - 5\lambda + 6 = 0 \implies \lambda = 2,\, 3$$

Homogeneous solution: $y_c = C_1e^{2t} + C_2e^{3t}$.

(b) **Nonhomogeneous Term:**

- $4e^{-2t}$: The exponential e^{-2t} does *not* appear in y_c, so no modification is needed.
- $3t^3$: Polynomial terms t^3, t^2, t, and constants are not part of y_c.

(c) The guess includes:

- A standalone Ae^{-2t} (no overlap)
- A cubic polynomial $Bt^3 + Ct^2 + Dt + E$ (no overlap)

Particular Solution:

$$\boxed{y_p = Ae^{-2t} + Bt^3 + Ct^2 + Dt + E}$$

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10. Suppose you wanted to use the method of undetermined coefficients to find a particular solution to

$$y'' - 2y' + y = 3te^t - t\sin(t).$$

What is an appropriate guess for the particular solution y_p?

Solution:

(a) **Homogeneous Solution:**

$$y'' - 2y' + y = 0 \implies \lambda^2 - 2\lambda + 1 = 0 \implies \lambda = 1\,(\text{repeated})$$

Homogeneous solution: $y_c = C_1e^t + C_2te^t$.

(b) **Nonhomogeneous Terms:**

- $3te^t$:
  - The term te^t already appears in y_c.
  - Standard modification: Multiply by t^k, where $k = 2$ (repeated root multiplicity).
  - Result: $t^2(At + B)e^t$.
- $-t\sin t$:
  - $\sin t$ and $\cos t$ do *not* appear in y_c.
  - Use polynomial multipliers for both $\sin t$ and $\cos t$: $(Ct+D)\cos t+(Et+F)\sin t$.

(c) The guess accounts for:

- A duplicated te^t term (resolved via t^2)
- First-degree polynomial-trigonometric combinations (no overlap)

Particular Solution:

$$\boxed{y_p = t^2(At + B)e^t + (Ct + D)\cos t + (Et + F)\sin t}$$

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11. Given that x^2 and x^{-1} are solutions to the corresponding homogeneous equation, find a particular solution to

$$x^2y'' - 2y = 3x^2 - 1, \quad x > 0.$$

Solution: nonconstant coefficients $\implies$ variation of parameters

Standard Form:

$$y'' - \frac{2}{x^2}y = 3 - \frac{1}{x^2} = r(x)$$

Wronskian Calculation:

$$W(x) = \begin{vmatrix} x^2 & x^{-1} \\ 2x & -x^{-2} \end{vmatrix} = x^2(-x^{-2}) - x^{-1}(2x) = -1 - 2 = -3$$

Particular Solution:

$$y_p(x) = u_1y_1 + u_2y_2$$

where

$$u_1 = \int -\frac{y_2 \cdot r}{W}\,dx, \qquad u_2 = \int \frac{y_1 \cdot r}{W}\,dx$$

$$\begin{aligned}
y_p &= -x^2 \int \frac{x^{-1}\left(3 - \frac{1}{x^2}\right)}{-3}dx + x^{-1}\int \frac{x^2\left(3 - \frac{1}{x^2}\right)}{-3}dx \\
&= \frac{x^2}{3}\int \left(3x^{-1} - x^{-3}\right)dx - \frac{1}{3}x^{-1}\int (3x^2 - 1)dx \\
&= x^2\left(\ln x + \frac{1}{6x^2}\right) - \frac{1}{3}x^{-1}(x^3 - x) \\
&= x^2 \ln x + \frac{1}{6} - \frac{x^2}{3} + \frac{1}{3} \\
&= x^2 \ln x + \frac{1}{2} - \frac{x^2}{3}
\end{aligned}$$

Notice that the term $-\frac{x^2}{3}$ is a homogeneous solution (since it is proportional to y_1) so we can leave it out of the particular solution

Final Solution:

$$\boxed{y_p(t) = x^2 \ln x + \frac{1}{2}}$$

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Analysis of ODEs

Review

- Where is a solution valid?
  - Solution is valid on a single interval where the solution is a function that is defined and differentiable.
- Existence and Uniqueness:
  - 1st order linear ODEs: If p and g are continuous on an interval $I = (a, b)$ containing the initial condition t_0, then the initial value problem

    $$y' + p(t)y = g(t), \quad y(t_0) = y_0$$

    has a unique solution on I.
  - 1st order nonlinear ODEs: Let the functions f and $\dfrac{\partial f}{\partial y}$ be continuous in some rectangle $(a, b) \times (c, d)$ containing the point (t_0, y_0). Then, there is a unique solution to the initial value problem

    $$y' = f(t, y), \quad y(t_0) = y_0$$

    on a sufficiently small interval $I_h = (t_0 - h, t_0 + h)$ around t_0.
  - 2nd order linear ODEs: Consider the initial value problem

    $$y'' + p(t)y' + q(t)y = g(t), \quad y(t_0) = y_0, \quad y'(t_0) = y_0'.$$

    If p, q, and g are continuous on an open interval $I = (a, b)$ that contains the point t_0, then there is exactly one solution to the initial value problem and the solution exists throughout the entire interval I.
- The Wronskian of y_1 and y_2 is defined by

  $$W[y_1, y_2](t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_1(t)y_2'(t) - y_1'(t)y_2(t).$$

- $\{y_1, y_2\}$ is a fundamental set of solutions means that the general solution is $c_1y_1 + c_2y_2$.
- Slope fields
- Equilibrium solutions
- Stability of equilibrium solutions:
  - (Asymptotically) stable: If you start near it, you go in towards it.
  - Unstable: If you start near it, you go away from it.
  - Semistable: If you start near on one side, you go towards it, but if you start near on the other side, you go away from it.
- Phase line diagrams

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12. Without solving the initial value problem, where is a unique solution guaranteed to exist?

$$y' - t^2\tan(t)y = \sqrt{4-t}, \quad y(0) = \pi.$$

Solution Interval:

- $\tan t$ discontinuous at $t = \frac{\pi}{2} + k\pi$ for $k = 0, \pm1, \pm2, \cdots$
- $\sqrt{4-t}$ requires $t \le 4$
- Largest interval containing $t = 0$: $\boxed{\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)}$

13. Without solving the initial value problem, where is a unique solution guaranteed to exist?

$$(t-1)w'' + w' - \ln(t+3)w = t^3\cos(t), \quad w(2) = -2, \quad w'(2) = 7.$$

Solution Interval:

$$\boxed{1 < t < \infty} \quad \text{(since } t = 1 \text{ makes coefficient zero and } t + 3 > 0 \text{ for } t > -3)$$

14. For which values t_0 and y_0 is the following initial value problem guaranteed to have a unique solution?

$$t^2y^2 - (t+y)y' = 0, \quad y(t_0) = y_0.$$

Unique Solution Exists When:

$$t_0 + y_0 \neq 0 \quad \text{(avoids denominator in } y' = \frac{t^2y^2}{t+y})$$

So all values on the line $y = -t$ cannot be initial conditions for the equation.

15. Show that x and xe^x form a fundamental set of solutions to

$$x^2y'' - x(x+2)y' + (x+2)y = 0, \quad x > 0.$$

Verify $y_1 = x$ is a solution

$$\begin{aligned} y_1 &= x \\ y_1' &= 1 \\ y_1'' &= 0 \end{aligned}$$

Substitute into the equation:

$$x^2(0) - x(x+2)(1) + (x+2)(x) = -x^2 - 2x + x^2 + 2x = 0$$

y_1 satisfies the differential equation.

Verify $y_2 = xe^x$ is a solution

$$\begin{aligned} y_2 &= xe^x \\ y_2' &= e^x + xe^x \\ y_2'' &= 2e^x + xe^x \end{aligned}$$

Substitute into the equation:

$$\begin{aligned} &x^2(2e^x + xe^x) - x(x+2)(e^x + xe^x) + (x+2)(xe^x) \\ &= 2x^2e^x + x^3e^x - x^2e^x - x^3e^x - 2xe^x - 2x^2e^x + x^2e^x + 2xe^x \\ &= (2x^2e^x - x^2e^x - 2x^2e^x + x^2e^x) + (x^3e^x - x^3e^x) + (-2xe^x + 2xe^x) \\ &= 0 \end{aligned}$$

y_2 satisfies the differential equation.

Wronskian:

$$W = \begin{vmatrix} x & xe^x \\ 1 & e^x(x+1) \end{vmatrix} = xe^x(x+1) - xe^x = x^2e^x$$

Non-zero for $x > 0 \implies$ Fundamental set.

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16. Solve for the explicit solution $u(x)$. Where is the solution to the initial value problem valid? How does this depend on a?

$$u' = u^2, \quad u(0) = a.$$

Solve the Differential Equation

This is a separable equation. Rearrange terms:

$$\frac{du}{u^2} = dx$$

Integrate both sides:

$$\int \frac{1}{u^2} du = \int 1\, dx$$

$$-\frac{1}{u} = x + C \quad \text{(General solution)}$$

where C is the constant of integration.

Apply Initial Condition $u(0) = a$

Substitute $x = 0$ and $u = a$:

$$-\frac{1}{a} = 0 + C \implies C = -\frac{1}{a}$$

The particular solution becomes:

$$-\frac{1}{u} = x - \frac{1}{a}$$

Solve for u:

$$u(x) = \frac{1}{\frac{1}{a} - x} = \frac{a}{1 - ax}$$

Domain of Validity

The solution $u(x) = \dfrac{a}{1 - ax}$ has restrictions based on the denominator:

- **Case** $a > 0$: Solution valid for $1 - ax > 0 \implies x < \dfrac{1}{a}$
- **Case** $a < 0$: Solution valid for $1 - ax > 0 \implies x > \dfrac{1}{a}$ (since dividing by negative reverses inequality)
- **Case** $a = 0$: Original equation becomes $u' = 0$ with constant solution $u(x) = 0$

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17. Consider the differential equation

$$f' = f(f-2)^2(f-4).$$

(a) Find the equilibrium solutions.

(b) Draw the phase line diagram.

(c) Sketch the slope field.

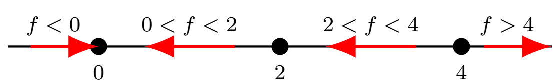
(d) Determine the stability of each equilibrium solution.

(e) Determine $\lim_{t\to\infty} f(t)$ for different initial values $f(0)$.

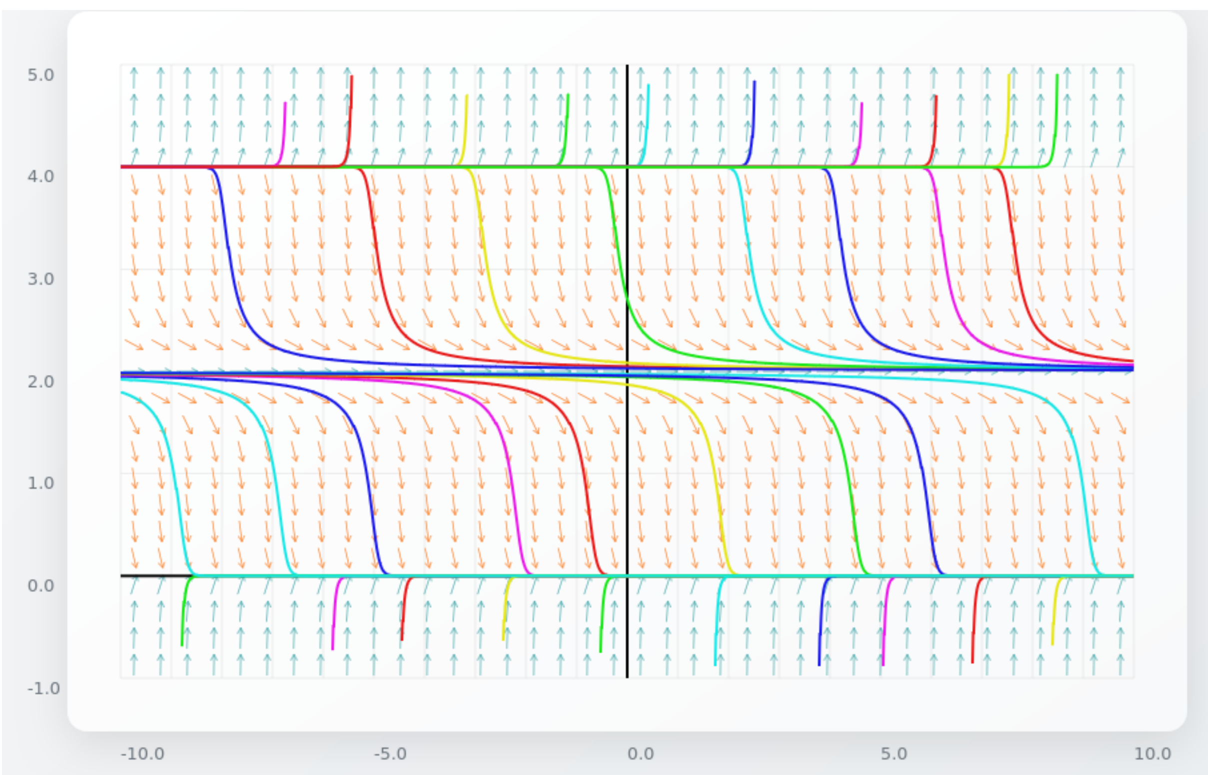
(a) Equilibrium Solutions

$$f = 0,\ 2,\ 4 \quad \text{(found by solving } f' = 0\text{)}$$

(b) Phase Line Diagram



(c) Slope Field



(d) Stability Classification

- $f = 0$: Asymptotically stable (arrows converge)
- $f = 2$: Semistable (converges from above, diverges below)
- $f = 4$: Unstable (arrows diverge)

(e) Limiting Behavior

- If $f(0) < 2$: $\lim_{t\to\infty} f(t) = 0$
- If $2 \leq f(0) < 4$: $\lim_{t\to\infty} f(t) = 2$
- If $f(0) \geq 4$: $\lim_{t\to\infty} f(t) = \infty$

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