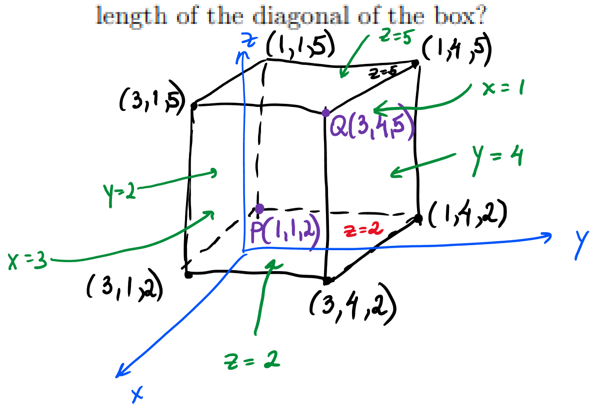
Math 251/221 WEEK in REVIEW 1. Fall 2024.

1. Draw a rectangular box that has the points $P(1,1,2)$ and $Q(3,4,5)$ as opposite vertices and has its faces parallel to the coordinate planes. Find the coordinates of the other six vertices of the box. What is the length of the diagonal of the box?



$$|PQ| = \sqrt{(3-1)^2+(4-1)^2+(5-2)^2} = \sqrt{2^2+3^2+3^2} = \sqrt{22}$$

a sphere center @ (a,b,c) of radius R

$$\boxed{(x-a)^2+(y-b)^2+(z-c)^2=R}$$

2. Find the center and radius of the sphere $x^2+y^2+z^2+4x+6y-10z+2=0$

$$(x^2+4x+\underbrace{4}_{2^2})+(y^2+6y+\underbrace{9}_{3^2})+(z^2-10z+\underbrace{25}_{5^2})+2=0+4+9+25$$

$$(x+2)^2+(y+3)^2+(z-5)^2=36$$

center @ $(-2,-3,5)$, radius is 6

3. Find an equation of the sphere given that it touches the yz- plane and has the center at $(2, 1, 3)$.

R = distance from $(2,1,3)$ to the yz-plane.

dist to yz dist to xz dist to xy

$R = 2$

Eqn. $(x-2)^2 + (y-1)^2 + (z-3)^2 = 2^2$ or $\boxed{(x-2)^2 + (y-1)^2 + (z-3)^2 = 4}$

4. Describe in words the region of $\mathbb{R}^3$ represented by the equation or inequality.

(a) $y = z$

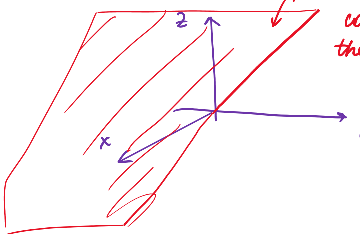
(b) $y > 2$

(c) $y^2 + z^2 \leq 4$

(d) $x^2 + y^2 + z^2 - 2z < 3$

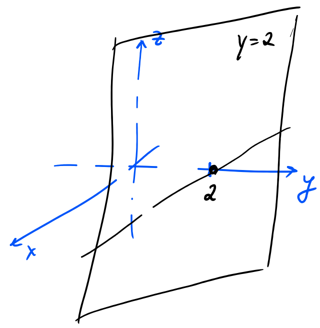
(a) $y = z$

plane $y=z$ contains the x-axis



(b) $y > 2$

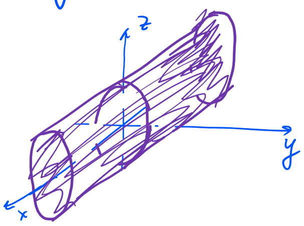
$y=2$ is a vertical plane perpendicular to the y-axis parallel to the xz-plane passes trough $(0,2,0)$



$y>2$ consists of points in $\mathbb{R}^3$ that lie to the right from the plane $y=2$. The plane is <u>not</u> included

(c) $\{y^2+z^2 \leq 4\}$ ← points inside the cylinder + the cylinder

a cylinder of radius 2 along the x-axis | $y^2+z^2=4$

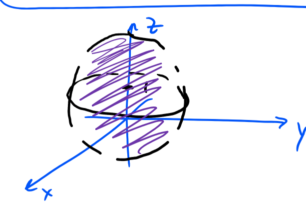


(d) $x^2+y^2+z^2-2z<3$

$x^2+y^2+(z^2-2z+1) = 3+1$

$x^2+y^2+(z-1)^2=4$

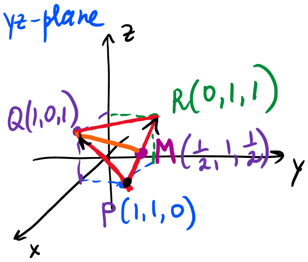
sphere with center $(0,0,1)$ of radius 2



$x^2+y^2+(z-1)^2<4$

consists of points inside the sphere, the sphere is <u>not</u> included.

5. Find the lengths of the sides and the medians of the triangle with the vertices $\underbrace{P(1,1,0)}_{\text{in xy-plane}}$, $\underbrace{Q(1,0,1)}_{\text{lies in the xz-plane}}$, and $\underbrace{R(0,1,1)}_{\text{in yz-plane}}$. Find the area of the triangle PQR.



$$\left.\begin{aligned}
|QR| &= \sqrt{(0-1)^2+(1-0)^2+(1-1)^2} = \sqrt{1^2+1^2} = \sqrt{2}\\
|PQ| &= \sqrt{(1-1)^2+(1-0)^2+(0-1)^2} = \sqrt{1^2+1^2} = \sqrt{2}\\
|PR| &= \sqrt{(0-1)^2+(1-1)^2+(1-0)^2} = \sqrt{1^2+1^2} = \sqrt{2}
\end{aligned}\right\} \text{sides.}$$

M is the midpoint for PR.

$$M\left(\frac{1+0}{2}, \frac{1+1}{2}, \frac{0+1}{2}\right) \text{ or } M\left(\frac{1}{2}, 1, \frac{1}{2}\right)$$

median $$|QM| = \sqrt{\left(\tfrac{1}{2}-1\right)^2+(1-0)^2+\left(\tfrac{1}{2}-1\right)^2} = \sqrt{\tfrac{1}{4}+1+\tfrac{1}{4}} = \sqrt{\tfrac{3}{2}}$$

Area $$A = \frac{1}{2}\left|\vec{PQ}\times\vec{PR}\right|$$

$$\vec{PQ} = \langle 1-1, 0-1, 1-0\rangle = \langle 0, -1, 1\rangle$$

$$\vec{PR} = \langle 0-1, 1-1, 1-0\rangle = \langle -1, 0, 1\rangle$$

$$\vec{PQ}\times\vec{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k}\\ 0 & -1 & 1\\ -1 & 0 & 1\end{vmatrix} = \vec{i}\begin{vmatrix}-1 & 1\\ 0 & 1\end{vmatrix} - \vec{j}\begin{vmatrix}0 & 1\\ -1 & 1\end{vmatrix} + \vec{k}\begin{vmatrix}0 & -1\\ -1 & 0\end{vmatrix}$$

$$= \vec{i}(-1+0) - \vec{j}(0-(-1)) + \vec{k}(0-1)$$

$$= -\vec{i} - \vec{j} - \vec{k}$$

$$A = \frac{1}{2}\left|\vec{PQ}\times\vec{PR}\right| = \frac{1}{2}\sqrt{(-1)^2+(-1)^2+(-1)^2} = \boxed{\frac{\sqrt{3}}{2}}$$

$\vec{b} = \langle 0, 3, -5 \rangle$

6. If $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{b} = 3\mathbf{j} - 5\mathbf{k}$, find:

   (a) A unit vector in the direction of $\mathbf{a}$.

   (b) A vector in the direction of $\mathbf{a} + \mathbf{b}$ with length 4.

(a) $\vec{a} = \langle 1, 2, -1 \rangle$

$|\vec{a}| = \sqrt{1^2 + 2^2 + (-1)^2} = \sqrt{6}$

unit vector $\quad \vec{u} = \dfrac{\vec{a}}{|\vec{a}|} = \dfrac{\langle 1, 2, -1 \rangle}{\sqrt{6}} = \dfrac{1}{\sqrt{6}} \langle 1, 2, -1 \rangle = \boxed{\left\langle \dfrac{1}{\sqrt{6}}, \dfrac{2}{\sqrt{6}}, -\dfrac{1}{\sqrt{6}} \right\rangle}$

(b) $\vec{a} + \vec{b} = \langle 1, 2, -1 \rangle + \langle 0, 3, -5 \rangle = \langle 1, 5, -6 \rangle$, $\quad |\vec{a} + \vec{b}| = \sqrt{1 + 5^2 + (-6)^2} = \sqrt{1 + 25 + 36} = \sqrt{62}$

a unit vector in the direction of $\vec{a} + \vec{b}$ is $\vec{u} = \dfrac{\vec{a} + \vec{b}}{|\vec{a} + \vec{b}|} = \dfrac{\langle 1, 5, -6 \rangle}{\sqrt{62}}$

a vector of length 4 in the direction of $\vec{a} + \vec{b}$ is $4\vec{u} = \boxed{\dfrac{4}{\sqrt{62}} \langle 1, 5, -6 \rangle}$

$\vec{F} = \langle 3, 2, -1 \rangle$, $|\vec{F}| = \sqrt{3^2+2^2+(-1)^2} = \sqrt{14}$

7. A constant force of $\mathbf{F} = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ moves an object along the line segment from $(1, 0, 2)$ to $(3, 4, 5)$. Find the work done. What is an angle between the force and the displacement vector?

$W = \vec{F} \cdot \vec{D}$

$\vec{D}$ is the displacement vector.

$\vec{D} = \langle 3-1, 4-0, 5-2 \rangle = \langle 2, 4, 3 \rangle$; $|\vec{D}| = \sqrt{2^2+4^2+3^2} = \sqrt{4+16+9} = \sqrt{29}$

$W = \langle 3, 2, -1 \rangle \cdot \langle 2, 4, 3 \rangle = 3(2) + 2(4) - 1(3) = 11$

an angle $\cos\theta = \dfrac{\vec{F} \cdot \vec{D}}{|\vec{F}| \cdot |\vec{D}|} = \dfrac{11}{\sqrt{14} \cdot \sqrt{29}}$

$\theta = \arccos\left(\dfrac{11}{\sqrt{14}\sqrt{29}}\right) \approx 57 \text{ (rad)}$

8. What restrictions must be made on b so that the vector $\underbrace{\overset{\langle 2,b,0\rangle}{2\mathbf{i} + b\mathbf{j}}}_{\vec{p}}$ is orthogonal to vector $\underbrace{\overset{\langle -3,2,1\rangle}{-3\mathbf{i} + 2\mathbf{j} + \mathbf{k}}}_{\vec{q}}$? to the vector $\mathbf{k}$?

Find b such that $\vec{p} \cdot \vec{q} = \underbrace{\langle 2, b, 0\rangle \cdot \langle -3, 2, 1\rangle}_{} = 0$

$$-6 + 2b + 0 = 0$$

$$2b = +6 \Rightarrow \boxed{b = +3}$$

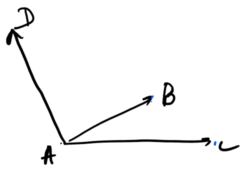
$\underbrace{\vec{p} = \langle 2, b, 0\rangle}_{\text{lies in the (xy) plane}}$ and $\underbrace{\vec{k} = \langle 0, 0, 1\rangle}_{\text{is perpendicular to all vectors in (xy) plane.}}$ — orthogonal for all b.

9. Find the scalar and vector projections of $\vec{n} = \langle 4, 2, 0\rangle$ onto $\vec{m} = \langle 1, 2, 3\rangle$.

scalar $\quad \text{comp}_{\vec{m}} \vec{n} = \dfrac{\vec{n}\cdot\vec{m}}{|\vec{m}|} = \dfrac{\langle 4,2,0\rangle \cdot \langle 1,2,3\rangle}{\sqrt{1^2+2^2+3^2}} = \dfrac{4+4}{\sqrt{14}} = \boxed{\dfrac{8}{\sqrt{14}}}$

vector $\quad \text{proj}_{\vec{m}} \vec{n} = \dfrac{\vec{n}\cdot\vec{m}}{|\vec{m}|^2}\,\vec{m} = \dfrac{8}{14}\langle 1,2,3\rangle = \boxed{\dfrac{4}{7}\langle 1,2,3\rangle}$

10. Given the points $A(1,0,1)$, $B(2,3,0)$, $C(-1,1,4)$, and $D(0,3,2)$. Find the volume of the parallelepiped with adjacent edges AB, AC, and AD



$V = \left| \vec{AB} \cdot (\vec{AC} \times \vec{AD}) \right|$ the absolute value of the scalar triple product.

$\vec{AB} = \langle 2-1, 3-0, 0-1 \rangle$

$\vec{AB} = \langle 1, 3, -1 \rangle$

$\vec{AC} = \langle -1-1, 1-0, 4-1 \rangle$

$\vec{AC} = \langle -2, 1, 3 \rangle$

$\vec{AD} = \langle 0-1, 3-0, 2-1 \rangle$

$\vec{AD} = \langle -1, 3, 1 \rangle$

$$\vec{AB} \cdot (\vec{AC} \times \vec{AD}) = \begin{vmatrix} 1 & 3 & -1 \\ -2 & 1 & 3 \\ -1 & 3 & 1 \end{vmatrix} \begin{matrix} 1 & 3 \\ -2 & 1 \\ -1 & 3 \end{matrix} = 1 - 9 + 6 - 1 - 9 + 6$$

$$= -18 + 12 = -6$$

$$V = |-6| = \boxed{6}$$