WIR#3

MATHEMATICS
TEXAS A&M UNIVERSITY

1. If $f(x) = x^2 + 2x - 4$, what is the average rate of change of $f(x)$ on the interval $[3, 8]$?

a b

$$AROC = \frac{f(b)-f(a)}{b-a} = \frac{f(8)-f(3)}{8-3} = \frac{\left((8)^2+2(8)-4\right)-\left(3^2+2(3)-4\right)}{5} = \frac{76-11}{5} = \frac{65}{5} = \boxed{13}$$

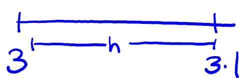
What is equivalent to AROC?

$$AROC = \frac{f(b)-f(a)}{b-a} = \frac{f(a+h)-f(a)}{h} = \text{Difference Quotient} = \text{slope of the secant line} = \text{Average velocity}$$

2. Let $f(x) = \dfrac{7}{x-5}$. Compute the difference quotient for $f(x)$ at $x = 3$ with $h = 0.1$.

a

$$\frac{f(a+h)-f(a)}{h} = \frac{f(3+0.1)-f(3)}{0.1} = \frac{\frac{7}{3.1-5} - \frac{7}{3-5}}{0.1} = \frac{\frac{-70}{19} - \left(\frac{-7}{2}\right)}{0.1} = \boxed{\frac{-35}{19}}$$



3. Let $f(x) = \sqrt{x+7}$. What is the instantaneous rate of change of $f(x)$ at $x = 2$?

a

$$IROC = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h}$$

Let's use the four-step process:

① $f(2+h) = \sqrt{2+h+7} = \sqrt{9+h}$

② $f(2+h) - f(2) = \sqrt{9+h} - \sqrt{2+7} = \sqrt{9+h} - 3$

③ $$\frac{f(2+h)-f(2)}{h} = \frac{\sqrt{9+h}-3}{h} \cdot \frac{(\sqrt{9+h}+3)}{(\sqrt{9+h}+3)} = \frac{\sqrt{9+h}\sqrt{9+h} \cancel{+3\sqrt{9+h}} \cancel{-3\sqrt{9+h}} - 3\cdot 3}{h(\sqrt{9+h}+3)}$$

$$= \frac{\cancel{9}+h\cancel{-9}}{h(\sqrt{9+h}+3)} = \frac{\cancel{h}}{\cancel{h}(\sqrt{9+h}+3)} = \frac{1}{\sqrt{9+h}+3}$$

④ $$\lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0} \frac{1}{\sqrt{9+h}+3} = \frac{1}{\sqrt{9+0}+3} = \boxed{\frac{1}{6}}$$

What is equivalent to IROC?

$$IROC = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \text{limit of the difference quotient} = \text{slope of the tangent line} = f'(a)$$

4. If $f(x) = \dfrac{x}{x+2}$,

$f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$ Let's use the four-step process!

(a) Find $f'(x)$.

① $f(x+h) = \dfrac{x+h}{x+h+2}$

Get common denominator!

② $f(x+h) - f(x) = \dfrac{x+h}{x+h+2} - \dfrac{x}{x+2} = \dfrac{(x+2)}{(x+2)}\cdot\dfrac{(x+h)}{(x+h+2)} - \dfrac{(x)}{(x+2)}\cdot\dfrac{(x+h+2)}{(x+h+2)}$

$= \dfrac{(x+2)(x+h) - x(x+h+2)}{(x+2)(x+h+2)}$

$= \dfrac{\cancel{x^2}+\cancel{xh}+\cancel{2x}+2h-\cancel{x^2}-\cancel{xh}-\cancel{2x}}{(x+2)(x+h+2)} = \dfrac{2h}{(x+2)(x+h+2)}$

③ $\dfrac{f(x+h)-f(x)}{h} = \dfrac{\dfrac{2h}{(x+2)(x+h+2)}}{h}$

$= \dfrac{2\cancel{h}}{(x+2)(x+h+2)}\cdot\dfrac{1}{\cancel{h}}$

$= \dfrac{2}{(x+2)(x+h+2)}$

④ $f'(x) = \lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\dfrac{2}{(x+2)(x+h+2)}$

$= \dfrac{2}{(x+2)(x+0+2)}$

$= \boxed{\dfrac{2}{(x+2)^2} = f'(x)}$

(b) What is the equation of the line tangent to the graph of $f(x)$ at $x = 1$?

① slope $= m = f'(1) = \dfrac{2}{(1+2)^2} = \dfrac{2}{9}$ ② Point → y-coordinate is $f(1) = \dfrac{1}{1+2} = \dfrac{1}{3}$

Eq of line w/ slope $= m = \frac{2}{9}$ and passes through $(1, 1/3)$

$y = mx + b$

$\frac{1}{3} = \frac{2}{9}(1) + b$

$b = \frac{1}{3} - \frac{2}{9} = \frac{1}{9}$

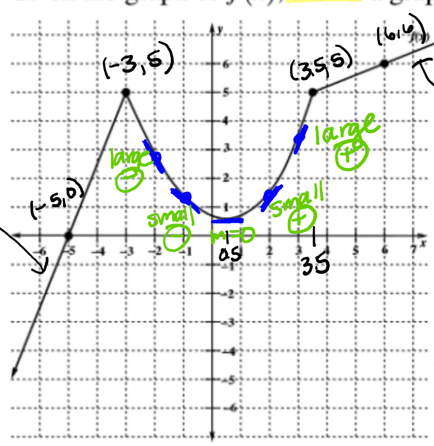
$\boxed{y = \frac{2}{9}x + \frac{1}{9}}$

5. Given the graph of $f(x)$, sketch a graph of $f'(x)$

Sketch!

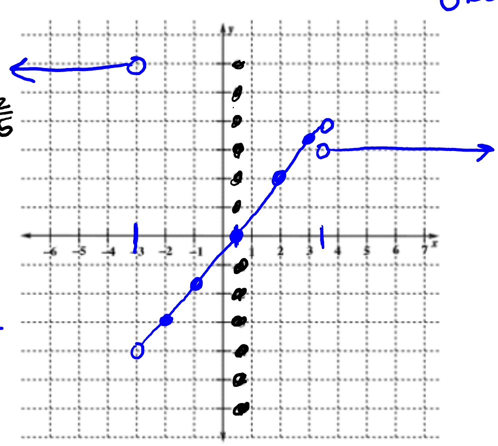
constant pos. slope

$m = \dfrac{5-0}{-3-(-5)} = \dfrac{5}{2}$



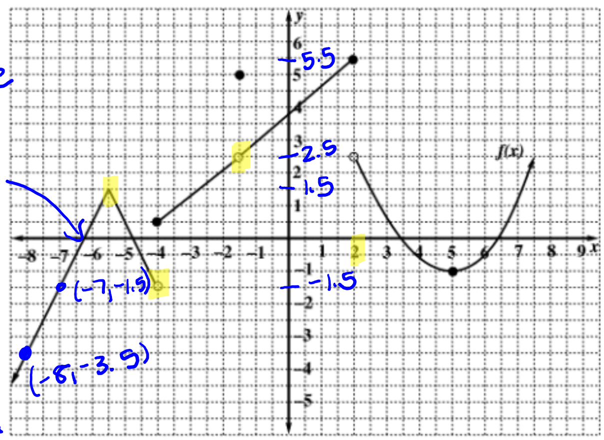
constant pos. slope

$m = \dfrac{6-5}{6-3.5} = \dfrac{1}{2.5} = \dfrac{2}{5}$



x	$f'(x)$
$x<-3$	$5/2$
-3	DNE
-2	large ⊖
-1	small ⊖
0.5	0
2	small ⊕
3	large ⊕
3.5	DNE
$x>3.5$	$2/5$

6. Use the graph of $f(x)$ to determine the following:



$f'(-6)$ is the slope of this line

$m = \frac{-3.5-(-1.5)}{-8-(-7)} = \frac{-3.5+1.5}{-8+7} = \frac{-2}{-1} = 2$

(a) $\lim_{x\to 2} f(x)$

$\lim_{x\to 2^-} f(x) = 5.5$

$\lim_{x\to 2^+} f(x) = 2.5$

$\Rightarrow \boxed{\lim_{x\to 2} f(x) \text{ DNE}}$

(b) $\lim_{x\to -1.5} f(x) = \boxed{2.5}$

(c) $\lim_{x\to -4^-} f(x) = \boxed{-1.5}$

(d) $\lim_{x\to -5.5} (2+[f(x)]^2) = \lim_{x\to 5.5} 2 + \left(\lim_{x\to -5.5} f(x)\right)^2$ ← Limit Laws from sec 1.2

$= 2 + (1.5)^2 = \boxed{4.25}$

(e) $f'(-6) = \boxed{2}$

(f) The value(s) of x for which $f(x)$ is discontinuous. Also, state which condition of the definition of continuity is the first to fail.

$\boxed{\begin{array}{l} x=-4 \rightarrow \text{II.} \\ x=-1.5 \rightarrow \text{III.} \\ x=2 \rightarrow \text{II.} \end{array}}$

$\lim_{x\to -4} f(x)$ DNE

$\lim_{x\to -1.5} f(x) \neq f(-1.5)$

$\lim_{x\to 2} f(x)$ DNE

Def. of Cont. at $x=a$:

I. $f(a)$ is defined

II. $\lim_{x\to a} f(x)$ exists

III. $\lim_{x\to a} f(x) = f(a)$

(g) the value(s) of x for which $f(x)$ is non-differentiable.

Discontinuous	Cusp or corner	Vertical Tangent Line
$x=-4$ $x=-1.5$ $x=2$	$x=-5.5$	None

7. Evaluate the limits. If the limit does not exist because of infinite behavior, describe the infinite behavior.

(a) $\lim_{x\to 4} \frac{2x^3-7x^2-4x}{x-4}$ $\left(\frac{0}{0}\right)$

$\lim_{x\to 4} (2x^3-7x^2-4x) = 2(4)^3-7(4)^2-4(4) = 0$

$\lim_{x\to 4} (x-4) = 4-4 = 0$

↳ Indeterminate Form ⇒ Do Algebra ⇒ Here, factor!

$\lim_{x\to 4} \frac{2x^3-7x^2-4x}{x-4} = \lim_{x\to 4} \frac{x(2x^2-7x-4)}{x-4} = \lim_{x\to 4} \frac{x(2x+1)\cancel{(x-4)}}{\cancel{x-4}} = \lim_{x\to 4} x(2x+1)$

$= 4(2\cdot 4+1)$

$= 4(9) = \boxed{36}$

(b) $\lim_{x\to 10} \underbrace{\frac{x+1}{x^2-20x+100}}_{f(x)}$ $\frac{\text{Nonzero \#}}{0}$

$\lim_{x\to 10}(x+1) = 10+1 = 11$

$\lim_{x\to 10}(x^2-20x+100) = 10^2-20(10)+100 = 0$

The limit does not exist, infinite limit, $x=10$ is a vertical asymptote

$\Rightarrow$ plug in value on either side of 10 to determine behavior:

$f(9.999) = 10999000 \Rightarrow \lim_{x\to 10^-} f(x) \to \infty$

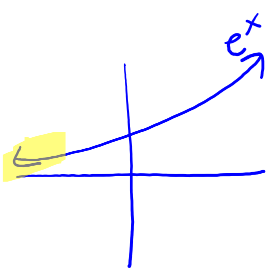
$f(10.001) = 11001000 \Rightarrow \lim_{x\to 10^+} f(x) \to \infty$

$\Rightarrow \boxed{\lim_{x\to 10} \frac{x+1}{x^2-20x+100} \to \infty}$

(c) $\lim_{x\to -\infty} \frac{5+e^{2x}-8e^{-x}}{e^x-4e^{-x}+9}$

Divide by the most negative e^{nx} in the denom.

$$= \lim_{x\to -\infty} \frac{\frac{5}{e^{-x}} + \frac{e^{2x}}{e^{-x}} - \frac{8e^{-x}}{e^{-x}}}{\frac{e^x}{e^{-x}} - \frac{4e^{-x}}{e^{-x}} + \frac{9}{e^{-x}}} = \lim_{x\to -\infty} \frac{5e^x + e^{3x} - 8}{e^{2x} - 4 + 9e^x} = \frac{-8}{-4} = \boxed{2}$$



$\lim_{x\to -\infty}(5\cancel{e^x}^{0} + \cancel{e^{3x}}^{0} - 8) = -8$

$\lim_{x\to -\infty}(\cancel{e^{2x}}^{0} - 4 + 9\cancel{e^x}^{0}) = -4$

8. Determine all vertical asymptote(s) and the location of all hole(s) of $f(x) = \frac{(x-3)^5\cancel{(x+2)}\cancel{(x-5)}}{(x-2)\cancel{(x-5)}(x+2)^{\cancel{2}}}$

① candidates for VA & holes: $(x-2)(x-5)(x+2)^2 = 0$

$x-2=0 \quad x-5=0 \quad x+2=0$

$x=2 \quad x=5 \quad x=-2$

② since $x-5$ is no longer in denom after cancelling

$\Rightarrow x-5=0$

$\boxed{x=5 \leftarrow \text{A hole at}}$

③ since $(x-2)$ and $(x+2)$ remain in denominator after cancelling

$\Rightarrow \boxed{x=2 \text{ \& } x=-2 \text{ are vertical asymptotes}}$

Our functions are continuous on their domains!

One exception: Piecewise Defined Functions!

9. Determine the interval(s) on which each of the following functions is continuous:

Recall Domain Restrictions:

① $\frac{f(x)}{g(x)} \Rightarrow g(x) \neq 0$

② $\sqrt[n]{f(x)}$ for any even n $\Rightarrow f(x) \geq 0$

③ $\log_b(f(x))$ for any base b $\Rightarrow f(x) > 0$

(a) $f(x) = \dfrac{x-5}{\sqrt{x+1}} + \ln(8-x)$

① $\sqrt{x+1} \neq 0$
$x+1 \neq 0$
$x \neq -1$

② $x+1 \geq 0$
$x \geq -1$

③ $8-x > 0$
$8 > x$
$x < 8$

$\boxed{(-1, 8)}$

(b) $f(x) = \begin{cases} \dfrac{x+1}{2x^2-3x-9} & \text{if } x < 1 \quad ⓐ \\ x^2+1 & \text{if } x \geq 1 \quad ⓑ \end{cases}$

Note: we are given a rule for all values of x.

① Check each rule (Domain Restrictions)

ⓐ $2x^2-3x-9 \neq 0$
$(2x+3)(x-3) \neq 0$
$2x+3 \neq 0$ $\quad$ $x-3 \neq 0$
$2x \neq -3$ $\quad$ ~~$x \neq 3$~~
$x \neq -3/2$ $\quad$ Not on int $x<1$

$\boxed{f(x) \text{ is discont. at } x = -3/2}$

ⓑ x^2+1
No Domain Restriction

② Check the definition of continuity at each cut-off # ($x=1$)

I. $f(1) = 1^2+1 = 2$ ✓

II. $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} \dfrac{x+1}{2x^2-3x-9} = \dfrac{1+1}{2(1)^2-3(1)-9} = \dfrac{2}{-10} = -\dfrac{1}{5}$

$\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} (x^2+1) = 1^2+1 = 2$

$\Rightarrow \lim\limits_{x \to 1} f(x)$ DNE

$\Rightarrow \boxed{f(x) \text{ is discont. at } x = 1}$

$\boxed{f(x) \text{ is continuous on } (-\infty, -3/2) \cup (-3/2, 1) \cup (1, \infty)}$