Math 308: Week-in-Review 12 (7.5 - 7.6)

7.5: Homogeneous Linear Systems with Constant Coefficients

Review

- How to solve a homogeneous linear system with constant coefficients (when you have distinct real eigenvalues)
  1. Assume your solution has the form $\mathbf{x}(t) = \boldsymbol{\xi} e^{rt}$.
  2. Plug this in to get an eigenvalue problem.
  3. Solve for the eigenvalues r_1 and r_2 and the corresponding eigenvectors $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$.
  4. The general solution is $c_1\boldsymbol{\xi}^{(1)}e^{r_1 t} + c_2\boldsymbol{\xi}^{(2)}e^{r_2 t}$.
- Phase plane/portrait: A phase plane/portrait is essentially a 2D version of the phase line. It shows you where the solution moves as time passes.
- An equilibrium point is a point where if you start there, you will remain there forever. The origin is always an equilibrium point of the differential equation system $\mathbf{x}' = A\mathbf{x}$.
- Stability of equilibrium points
  - Asymptotically stable: If you start near the equilibrium point, you will be sucked into it as $t \to \infty$.
  - Stable: If you start near the equilibrium point, you will stay near it.
  - Unstable: There is at least one point near the equilibrium point that goes away from the equilibrium point.

7.6: Complex Eigenvalues

Review

- To solve the system $\mathbf{x}' = A\mathbf{x}$ when you have complex eigenvectors:
  - Solve for just one of the eigenvectors.
  - Separate $\boldsymbol{\xi} e^{rt}$ into its real and imaginary parts.
  - The real and imaginary parts form a fundamental set of solutions.
  - (Assuming that A is 2×2. If A is larger, then there are also more solutions.)

1. Find the general solution, sketch the phase plane, and determine the stability of the equilibrium point at the origin.

$$\mathbf{x}' = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} \mathbf{x}$$

2. Solve the initial value problem when $\mathbf{x}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Draw this solution on the phase plane and sketch the graph of $x_1(t)$ and $x_2(t)$.

3. Find the general solution, sketch the phase plane, and determine the stability of the equilibrium point at the origin.

$$x_1' = -5x_1 + 4x_2$$
$$x_2' = \frac{3}{2}x_1 - 4x_2$$

4. Find the general solution, sketch the phase plane, and determine the stability of the equilibrium point at the origin.

$$x' = x + 2y$$
$$y' = -2x + 5y$$

5. Find the general solution, sketch the phase plane, and determine the stability of the equilibrium point at the origin.

$$x' = 3x + y$$
$$y' = -2x + y$$

6. Find the general solution, sketch the phase plane, and determine the stability of the equilibrium point at the origin.

$$\mathbf{x}' = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} \mathbf{x}$$

7. Find the general solution, sketch the phase plane, and determine the stability of the equilibrium point at the origin. Solve the initial value problem with $\mathbf{x}(0) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

$$\mathbf{x}' = \begin{bmatrix} 1 & -8 \\ 1 & -3 \end{bmatrix} \mathbf{x}$$

8. Classify the types and stability of the equilibrium point(s) of the system

$$x' = \begin{pmatrix} \alpha - 1 & \alpha + 1 \\ -2/3 & 1/3 \end{pmatrix} x$$

for different values of the parameter α.