Math 308: Week-in-Review 11 (7.1 - 7.4)

7.2: Review of Matrices

Review

- A matrix A is a rectangular table of numbers. We typically denote the matrix with a capital letter and its entries with lower case letters. For example, the entry in the 3rd row and 2nd column of A would be denoted a_{32}.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

- The identity matrix is denoted I and is the matrix that is all zeros, except for along the top-left to bottom-right diagonal, where the entries are all 1. It can have any size but it must be square. For example, the 2×2 and 3×3 cases are

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

- A vector is just a matrix with either one row or one column.

Basic Matrix Operations

- To add or subtract two matrices, add or subtract the corresponding entries.
- To multiply a matrix times a scalar (scalar = number), multiply each entry of the matrix by the scalar.

1. (a) Compute:

$$\overset{(2\times 3)}{\begin{bmatrix} 3 & -2 & 1 \\ 2 & 5 & -3 \end{bmatrix}} + \overset{(2\times 3)}{\begin{bmatrix} 5 & 1 & 7 \\ -1 & 6 & 4 \end{bmatrix}} = \checkmark$$

$$\begin{bmatrix} 8 & -1 & 8 \\ 1 & 11 & 1 \end{bmatrix}$$

* add corresponding entries

(b) Compute:

$$\overset{(3\times 3)}{\begin{bmatrix} 8 & 4 & -1 \\ 5 & -3 & 2 \\ 1 & 3 & -5 \end{bmatrix}} - \overset{(3\times 3)}{\begin{bmatrix} 2 & -1 & 2 \\ 1 & -2 & 1 \\ 8 & 2 & -3 \end{bmatrix}} = \checkmark$$

$$\begin{bmatrix} 6 & 5 & -3 \\ 4 & -1 & 1 \\ -7 & 1 & -2 \end{bmatrix}$$

* subtract corresponding entries

(c) Compute:

$$-3\begin{bmatrix} 3 & 7 \\ -2 & 4 \end{bmatrix} =$$

* scalar multiplication
* multiply all elements by scalar

$$\begin{bmatrix} -9 & -21 \\ 6 & -12 \end{bmatrix}$$

Matrix Multiplication

- To multiply two matrices together, you multiply the rows of the first matrix with the columns of the second matrix.

1. (a) Compute:

$2 \times \boxed{2 \quad 2} \times 3$ ✓ $\quad 2 \times 3$

$$\begin{bmatrix} 4 & -2 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} -2 & 3 & 1 \\ 4 & 1 & 2 \end{bmatrix} = \begin{bmatrix} -16 & 10 & 0 \\ -22 & 9 & 13 \end{bmatrix}$$

$$\begin{bmatrix} (4)(-2)+(-2)(4) & (4)(3)+(-2)(1) & (4)(1)+(-2)(2) \\ (1)(-2)+(6)(4) & (1)(3)+(6)(1) & (1)(1)+(6)(2) \end{bmatrix}$$

(b) Compute:

$2 \times \boxed{3 \quad 3} \times 2$ ✓ $\quad 2 \times 2$

$$\begin{bmatrix} 5 & 1 & -3 \\ 7 & 2 & -1 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 2 & 5 \\ -6 & 1 \end{bmatrix} = \begin{bmatrix} 40 & 7 \\ 38 & 16 \end{bmatrix}$$

$$\begin{bmatrix} (5)(4)+(1)(2)+(-3)(-6) & (5)(1)+(1)(5)+(-3)(1) \\ (7)(4)+(2)(2)+(-1)(-6) & (7)(1)+(2)(5)+(-1)(1) \end{bmatrix}$$

(c) Compute:

$3 \times \boxed{3 \quad 3} \times 1$ ✓ $\quad 3 \times 1$

$$\begin{bmatrix} 6 & -1 & 2 \\ 8 & 1 & 1 \\ -1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 24 \\ 24 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} (6)(3)+(-1)(-2)+(2)(2) \\ (8)(3)+(1)(-2)+(1)(2) \\ (-1)(3)+(3)(-2)+(4)(2) \end{bmatrix}$$

Trace

- The trace of a matrix is the sum of its diagonal elements (the top-left to bottom-right diagonal).

1. Compute the trace of the following matrices.

(a) $\operatorname{tr}\left(\begin{bmatrix} 3 & -8 \\ 5 & -6 \end{bmatrix}\right) = (3)+(-6) = 3-6 = \boxed{-3}$

(b) $\operatorname{tr}\left(\begin{bmatrix} 4 & -2 & 1 \\ 8 & 1 & 2 \\ 1 & -1 & -2 \end{bmatrix}\right) = 4+1-2 = \boxed{3}$

Determinant

- The determinant of a 2×2 matrix is

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

- The determinant of a 3×3 matrix is (expansion along the first row):

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}, = a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

1. Compute the following determinants.

(a) $\begin{vmatrix} 4 & 1 \\ -2 & 4 \end{vmatrix} = (4)(4)-(-2)(1) = 16+2 = \boxed{18}$

(b) $\begin{vmatrix} -1 & 2 \\ 5 & 8 \end{vmatrix} = (-1)(8)-(5)(2) = -8-10 = \boxed{-18}$

(c) $\begin{vmatrix} 2 & 2 & 3 \\ -1 & 6 & 2 \\ 1 & -2 & 4 \end{vmatrix} = 2\begin{vmatrix} 6 & 2 \\ -2 & 4 \end{vmatrix} - 2\begin{vmatrix} -1 & 2 \\ 1 & 4 \end{vmatrix} + 3\begin{vmatrix} -1 & 6 \\ 1 & -2 \end{vmatrix} = \boxed{56}$

$= (2)(28) - 2(-6) + 3(-4)$

Writing a System of Linear Equations in Matrix Form

- A common use of matrices and vectors is that we can use them to write a system of linear equations into a compact form.

1. Multiply out:

$$\begin{bmatrix} 3 & -2 & 4 \\ 1 & 3 & 5 \\ 8 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_1 - 2x_2 + 4x_3 \\ x_1 + 3x_2 + 5x_3 \\ 8x_1 + x_2 + 2x_3 \end{bmatrix}$$

1. Write the following systems of equations into matrix-vector form.

   (a)

coeffs of x_1 | coeffs of x_2 | right hand side

$$\begin{aligned} 4x_1 - 2x_2 &= 6 \\ 5x_1 + 2x_2 &= 4 \end{aligned} = \begin{bmatrix} 4 & -2 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

unknown variables

   (b)

$$\begin{aligned} 5x_1 - 2x_2 + x_3 &= 1 \\ 4x_2 + 3x_3 &= 5 \\ -2x_2 &= 1 \end{aligned} \quad \begin{bmatrix} 5 & -2 & 1 \\ 0 & 4 & 3 \\ 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}$$

   (c)

coeffs of x_1 | coeffs of x_2 | coeffs of x_3

$$\begin{aligned} x_3 - 2x_2 &= 6 \\ 2x_2 + x_1 &= 5 \\ 3x_1 - 2x_3 + x_2 &= 2 \end{aligned} \quad \begin{bmatrix} 0 & -2 & 1 \\ 1 & 2 & 0 \\ 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix}$$

7.1: Introduction to Systems of 1st-Order ODEs

Review

- A system of differential equations is just a few differential equations. The interesting part is that they might be coupled (that is, the solution to one equation depends on the solution to another one). For example,

$$u' = 3v + u$$
$$v' = -v + 4u$$

- A system of first-order differential equations is linear if they can be written in the form

$$\begin{aligned} x_1' &= p_{11}(t)x_1 + \cdots + p_{1n}(t)x_n + g_1(t) \\ x_2' &= p_{21}(t)x_1 + \cdots + p_{2n}(t)x_n + g_2(t) \\ &\vdots \\ x_n' &= p_{n1}(t)x_1 + \cdots + p_{nn}(t)x_n + g_n(t) \end{aligned}$$

  In matrix form, it looks like:

$$\mathbf{x}' = P(t)\mathbf{x} + \mathbf{g}(t)$$

- A system of linear first-order differential equations is homogeneous if the g_i terms are all 0 (i.e., if $\mathbf{g}(t) = \mathbf{0}$).

- A system of differential equations can also have initial conditions. For example, the above system could have the initial conditions

$$x_1(0) = a_1, \quad x_2(0) = a_2, \quad \ldots, \quad x_n(0) = a_n,$$

  which can also be written as $\mathbf{x}(0) = \mathbf{a}$.

- You can convert a higher order differential equation into a system of first-order differential equations.

1. For the following systems of ODEs, determine if they are linear or nonlinear. If they are linear, then also determine if they are homogeneous or nonhomogeneous. If it is linear, write it into matrix-vector form.

(a)

$$x_1 - 2x_2 = 4t$$
$$x_1^2 + x_2 = \cos(t)$$

non-linear

(b)

linear, non-homogeneous

$$2t^2x_1 + x_2 - 2tx_3 = \sin(t)$$
$$x_1 - 2x_3 = 0$$
$$x_2 + \sin(t)x_3 = 0$$

$$\begin{bmatrix} 2t^2 & 1 & -2t \\ 1 & 0 & -2 \\ 0 & 1 & \sin(t) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \sin(t) \\ 0 \\ 0 \end{bmatrix}$$

(c)

$$x_1 - x_2x_3 = 6$$
$$x_2 - \sin(x_3) = 5t$$
$$2x_1 + (t^5 + 3)x_3 = 0$$

non-linear

(d)

linear, homogeneous

$$\sin(t)x_1 - x_3 = 0$$
$$x_1 - 4\ln(t)x_2 = 4tx_3 \rightarrow x_1 - 4\ln(t)x_2 - 4tx_3 = 0$$
$$x_2 + 2x_3 = 0$$

$$\begin{bmatrix} \sin(t) & 0 & -1 \\ 1 & -4\ln(t) & -4t \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

1. Suppose we have two tanks. Tank 1 holds 12 L and Tank 2 holds 18 L.

   - Salt water containing 3 g/L of salt is flowing into Tank 1 at a rate of 6 L/min.
   - Salt water containing 1 g/L of salt is flowing into Tank 2 at a rate of 8 L/min.
   - Salt water flows out of Tank 1 at a rate of 7 L/min, of which 3 L/min flows into Tank 2 and the rest leaves the system.
   - Salt water flows out of Tank 2 at a rate of 10 L/min, of which 2 L/min flows into Tank 1 and the rest leaves the system.

   Both tanks initially start with 50 g of salt. Write down an initial value problem that models this system. Write your answer into matrix-vector form.

$* \; V_1 = 12 + [(6+2)-7]t$

$* \; V_2 = 18 + [(8+3)-10]t$

Let $x_1(t) \to$ amount of salt in tank 1 at time t

$x_2(t) \to$ amount of salt in tank 2 at time t

$$\frac{dx_1}{dt} = \underbrace{(3\ g/L)(6\ L/min)}_{\text{rate in}} + \underbrace{\left(\frac{x_2\, g}{18+t}\right)\left(\frac{2\ L}{min}\right)}_{\text{from tank 2}} - \underbrace{\left(\frac{x_1\, g}{12+t}\right)\left(\frac{7\ L}{min}\right)}_{\text{rate out}} = 18 + \frac{2x_2}{18+t} - \frac{7x_1}{12+t}$$

$$\frac{dx_2}{dt} = \underbrace{(1\ g/L)(8\ L/min)}_{\text{rate in}} + \underbrace{\left(\frac{x_1\, g}{12+t}\right)\left(\frac{3\ L}{min}\right)}_{\text{from tank 1}} - \underbrace{\left(\frac{x_2\, g}{18+t}\right)\left(\frac{10\ L}{min}\right)}_{\text{rate out}} = 8 + \frac{3x_1}{12+t} - \frac{10x_2}{18+t}$$

$$\frac{dx_1}{dt} = \frac{-7x_1}{12+t} + \frac{2x_2}{18+t} + 18, \quad \frac{dx_2}{dt} = \frac{3x_1}{12+t} - \frac{10x_2}{18+t} + 8, \quad x_1(0) = 50, \; x_2(0) = 50$$

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} \frac{-7}{12+t} & \frac{2}{18+t} \\ \frac{3}{12+t} & \frac{-10}{18+t} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 18 \\ 8 \end{bmatrix}, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 50 \\ 50 \end{bmatrix}}$$

1. Suppose we initially have 120 rabbits that reproduce at a rate proportional to the current population. There are 12 wolves that live nearby. The wolves eat the rabbits at a rate proportional to the product of the population of rabbits and the population of wolves. The wolves reproduce at a rate proportional to the number of rabbits they eat. Write down an initial value problem that models this system.

$r = r(t) \rightarrow$ population of rabbits at time t

$w = w(t) \rightarrow$ population of wolves at time t

$r(0) = 120, \quad w(0) = 12$

$$\begin{cases} \dfrac{dr}{dt} = \alpha r - \beta r w \\[2ex] \dfrac{dw}{dt} = \delta r w \end{cases}$$

α = rabbits growth rate

β = rate of predation

δ = wolves growth rate

1. Write the following differential equation as a system of differential equations.

$$y'' - 4y' + 5y = \sin(t)$$

$$y_1 = y, \quad y_2 = y_1' = y' \Rightarrow y_2' = y_1' = y''$$

$$\begin{cases} y_1' = y_2 \\ y_2' = -5y_1 + 4y_2 + \sin(t) \end{cases} \Rightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -5 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \sin(t) \end{bmatrix}$$

1. Write the following initial value problem as a system of differential equations.

$$y''' - t^3 y'' + \cos(t) y' - y = 5t, \quad y(0) = 2, \quad y'(0) = 5, \quad y''(0) = 1$$

$$y_1 = y, \quad y_2 = y_1' = y', \quad y_3 = y_2' = y''$$

$$\begin{cases} y_1' = y_2 \\ y_2' = y_3 \\ y_3' = y_1 - \cos(t) y_2 + t^3 y_3 + 5t \end{cases} \Rightarrow \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -\cos(t) & t^3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 5t \end{bmatrix}$$

$$\begin{bmatrix} y_1(0) \\ y_2(0) \\ y_3(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$$

1. Write the following initial value problem as a system of differential equations.

$$z^{(4)} - tz'' + z = 6t^2 + \sin(t), \quad z(1) = 2, \quad z'(1) = -2, \quad z''(1) = 5, \quad z'''(1) = 3$$

$$z_1 = z, \quad z_2 = z_1' = z', \quad z_3 = z_2' = z'', \quad z_4 = z_3' = z'''$$

$$\begin{cases} z_1' = z_2 \\ z_2' = z_3 \\ z_3' = z_4 \\ z_4' = -z_1 + t z_3 + 6t^2 + \sin(t) \end{cases} \rightsquigarrow \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & t & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 6t^2 + \sin(t) \end{bmatrix}$$

$$\begin{bmatrix} z_1(1) \\ z_2(1) \\ z_3(1) \\ z_4(1) \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 5 \\ 3 \end{bmatrix}$$

7.3: Linear Independence, Eigenvalues, and Eigenvectors

Review

- Two vectors are linearly dependent if one is a scalar multiple of the other.
- More generally, n vectors $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}$ are linearly dependent if there exist constants $c_1, \dots, c_n$ such that
$$c_1\mathbf{x}^{(1)} + \cdots + c_n\mathbf{x}^{(n)} = \mathbf{0}.$$
- If a set of vectors is not linearly dependent, then we say they are linearly independent.
- λ is an eigenvalue and ξ is a corresponding eigenvector of A if
$$A\xi = \lambda\xi.$$
- How to find eigenvalues/vectors:
  1. Find the characteristic equation: $\lambda^2 - \text{tr}(A)\lambda + \det(A) = 0$.
  2. Solve the characteristic equation to find the eigenvalues.
  3. Nonzero solutions to $(A - \lambda I)\xi = 0$ are the eigenvectors.

1. For each pair of vectors, are they linearly dependent or independent? Note: $\begin{bmatrix}3\\-6\end{bmatrix} = -3\begin{bmatrix}-1\\2\end{bmatrix}$

$\hookrightarrow 1\cdot\begin{bmatrix}3\\-6\end{bmatrix} + 3\begin{bmatrix}-1\\2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

(a) $\left\{ \begin{bmatrix}3\\-6\end{bmatrix}, \begin{bmatrix}-1\\2\end{bmatrix} \right\}$

$\det\begin{bmatrix}3 & -1\\-6 & 2\end{bmatrix} = (3)(2) - (-6)(-1) = 6 - 6 = 0$

linearly dependent

(b) $\left\{ \begin{bmatrix}4\\6\end{bmatrix}, \begin{bmatrix}0\\-3\end{bmatrix} \right\}$

$\det\begin{bmatrix}4 & 0\\6 & -3\end{bmatrix} = (4)(-3) + (6)(0) = -12 \neq 0$

linearly independent

(c) $\left\{ \begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}2\\4\end{bmatrix} \right\}$

$\det\begin{bmatrix}0 & 2\\0 & 4\end{bmatrix} = (0)(4) - (0)(2) = 0$

linearly dependent

Note: $c\cdot\begin{bmatrix}0\\0\end{bmatrix} + 0\cdot\begin{bmatrix}2\\4\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$ for any $c \neq 0$

(d) $\left\{ \begin{bmatrix}3\\-5\\-1\end{bmatrix}, \begin{bmatrix}2\\4\\6\end{bmatrix} \right\}$

$c_1\begin{bmatrix}3\\-5\\-1\end{bmatrix} + c_2\begin{bmatrix}2\\4\\6\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \Rightarrow$
$3c_1 + 2c_2 = 0$ (*)
$-5c_1 + 4c_2 = 0$ (**)
$-c_1 + 6c_2 = 0$ (***)

(***) & (*) Solve for c_1, c_2

$\begin{bmatrix}3 & 2 & 0\\-1 & 6 & 0\end{bmatrix} \xrightarrow{3*\text{row }2} \begin{bmatrix}3 & 2 & 0\\-3 & 18 & 0\end{bmatrix} \xrightarrow[\text{to row 2}]{\text{add row 1}} \begin{bmatrix}3 & 2 & 0\\0 & 20 & 0\end{bmatrix} \Rightarrow 20c_2 = 0$

$\Rightarrow c_2 = 0$

$3c_1 + 2c_2 = 0$

$\Rightarrow c_1 = 0$

linearly independent

1. Find the eigenvalues and eigenvectors of the matrix $\begin{bmatrix} 6 & -2 \\ 4 & 2 \end{bmatrix}$.

$$\det(A-\lambda I) = \det\begin{pmatrix} 6-\lambda & -2 \\ 4 & 2-\lambda \end{pmatrix}$$

$$= (6-\lambda)(2-\lambda)+8$$

$$= \lambda^2 - 8\lambda + 20$$

$$= 0$$

Eigenvalues

$$\boxed{\lambda_1 = 4+2i}, \quad \boxed{\lambda_2 = 4-2i}$$

$$\lambda = \frac{8 \pm \sqrt{8^2 - 4.1.20}}{2} = \frac{8 \pm \sqrt{-16}}{2} = \frac{8 \pm 4i}{2} = 4 \pm 2i$$

Eigenvectors: $\lambda_1 = 4+2i$

$$(A-\lambda_1 I)v_1 = 0 \Rightarrow \begin{pmatrix} 6-(4+2i) & -2 \\ 4 & 2-(4+2i) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 2-2i & -2 \\ 4 & -2-2i \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow 2(1-i)x_1 - 2x_2 = 0 \Rightarrow (1-i)x_1 = x_2$$

Set $x_1 = 1$. Then $\boxed{v_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1-i \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} - i\begin{pmatrix} 0 \\ 1 \end{pmatrix}}$

$\lambda_2 = 4-2i$

$$\begin{pmatrix} 6-(4-2i) & -2 \\ 4 & 2-(4-2i) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 2+2i & -2 \\ 4 & -2+2i \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow 2(1+i)x_1 - 2x_2 = 0 \Rightarrow (1+i)x_1 = x_2$$

Set $x_1 = 1$. Then $\boxed{v_2 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1+i \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + i\begin{pmatrix} 0 \\ 1 \end{pmatrix}}$

1. Find the eigenvalues and eigenvectors of the matrix $\underbrace{\begin{bmatrix} 4 & -1 \\ 5 & -2 \end{bmatrix}}_{A}$.

$\operatorname{tr}(A) = 4 - 2 = 2$

$\det(A) = (4)(-2) - (5)(-1)$
$= -8 + 5$
$= -3$

* characteristic polynomial *

$\lambda^2 - \operatorname{tr}(A)\lambda + \det(A) = 0$

$\Rightarrow \lambda^2 - 2\lambda - 3 = 0$

$(\lambda - 3)(\lambda + 1) = 0$

$\boxed{\lambda_1 = 3, \ \lambda_2 = -1}$ Eigenvalues

Eigenvectors

$\lambda_1 = 3$: $(A - 3I)v_1 = 0 \Rightarrow \begin{pmatrix} 4-3 & -1 \\ 5 & -2-3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 1 & -1 \\ 5 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$x_1 - x_2 = 0 \Rightarrow x_1 = x_2$

* choose $x_1 = x_2 = 1$

$\boxed{v_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$

$\lambda_2 = -1$: $(A + I)v_2 = 0 \Rightarrow \begin{pmatrix} 4+1 & -1 \\ 5 & -2+1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 5 & -1 \\ 5 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow 5x_1 - x_2 = 0$

$\Rightarrow 5x_1 = x_2$

* choose $x_1 = 1 \Rightarrow x_2 = 5$

$\boxed{v_2 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}}$

1. Find the eigenvalues and eigenvectors of the matrix $\underset{A}{\begin{bmatrix} 2 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix}}$.

$tr(A) = 2-2 = 0$

$\det(A) = -4-2 = -6$

characteristic polynomial

$\lambda^2 - tr(A)\lambda + \det(A) = 0$

Eigenvalues

$\Rightarrow \lambda^2 - 6 = 0 \Rightarrow \lambda^2 = 6 \Rightarrow \lambda = \pm\sqrt{6} \Rightarrow \lambda_1 = \sqrt{6}$, $\boxed{\lambda_2 = -\sqrt{6}}$

Eigenvectors

$\lambda_1 = \sqrt{6}$: $(A - \sqrt{6}I)v_1 = \begin{pmatrix} 2-\sqrt{6} & \sqrt{2} \\ \sqrt{2} & -2-\sqrt{6} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$(2-\sqrt{6})x_1 + \sqrt{2}x_2 = 0$

$\sqrt{2}x_2 = (\sqrt{6}-2)x_1$

* choose $x_1 = 1 \Rightarrow x_2 = \dfrac{\sqrt{6}-2}{\sqrt{2}} = \sqrt{3} - \sqrt{2}$

$$\boxed{v_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ \sqrt{3}-\sqrt{2} \end{pmatrix}}$$

$\lambda_2 = -\sqrt{6}$: $(A + \sqrt{6}I)v_2 = \begin{pmatrix} 2+\sqrt{6} & \sqrt{2} \\ \sqrt{2} & -2+\sqrt{6} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\sqrt{2}x_1 + (-2+\sqrt{6})x_2 = 0$

* choose $x_2 = 1 \Rightarrow x_1 = \dfrac{2-\sqrt{6}}{\sqrt{2}} = \sqrt{2} - \sqrt{3}$

$$\boxed{v_2 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \sqrt{2}-\sqrt{3} \\ 1 \end{pmatrix}}$$