$\sum_{n=1}^{\infty} a_n$, $a_n = f(n)$, positive, decreasing and continuous on $[1,\infty)$
$\sum_{n=1}^{\infty} f(n)$ and $\int_1^{\infty} f(x)dx$ both converge or both diverge.

1. (a) Use the Integral test to determine whether or not the series $\boxed{\sum_{n=1}^{\infty} n^2 e^{-n^3}}$ is convergent.

(a) $\sum_{n=1}^{\infty} n^2 e^{-n^3}$, $n^2 e^{-n^3}$ – positive on $[1,\infty)$
continuous on $[1,\infty)$
decreasing on $[1,\infty)$

$f(x) = x^2 e^{-x^3}$ $\qquad e^{-x^3} > 0$

$f'(x) = 2xe^{-x^3} + x^2(-3x^2)e^{-x^3} = e^{-x^3}(2x - 3x^2) < 0$

$2x - 3x^2 < 0$

$x(2-3x) < 0$ $\qquad$ $0 \quad 3/2$

negative at least on $(0,\infty)$

$f'(x) < 0 \rightarrow f(x)$ is decreasing.

$$\int_1^{\infty} x^2 e^{-x^3} dx = \lim_{N\to\infty} \int_1^N x^2 e^{-x^3} dx$$

$u = -x^3$
$du = -3x^2 dx$ or $x^2 dx = -\frac{du}{3}$
$x=1 \longrightarrow u=-1$
$x=N \longrightarrow u=-N^3$

$$= \lim_{N\to\infty} \int_{-1}^{-N^3} e^u\left(-\frac{du}{3}\right) = -\frac{1}{3}\lim_{N\to\infty} e^u\Big|_{-1}^{-N^3} = -\frac{1}{3}\left[\cancel{\lim_{N\to\infty} e^{-N^3}}^{\,0} - e^{-1}\right] = \frac{1}{3e}$$ hence convergent.

By the Integral Test, $\sum_{n=1}^{\infty} n^2 e^{-n^3}$ also converges.

(b) Approximate the sum of the series $\sum_{n=1}^{\infty} n^2 e^{-n^3}$ by using the sum of first 4 terms. Estimate the error involved in this approximation.

$S = S_n + R_n$
S_n is the n-th partial sum
R_n is the remainder.

$$\boxed{\int_{n+1}^{\infty} f(x)dx \le R_n \le \int_n^{\infty} f(x)dx}$$

$S_4 = a_1 + a_2 + a_3 + a_4 = e^{-1} + 4e^{-8} + 9e^{-27} + 16e^{-64}$

$$\int_5^{\infty} x^2 e^{-x^3} dx \le R_4 \le \int_4^{\infty} x^2 e^{-x^3} dx$$

$$\int_a^{\infty} x^2 e^{-x^3} dx = \frac{1}{3e^{a^3}} = \frac{e^{-a^3}}{3}$$

$$\frac{1}{3}e^{-125} \le R_4 \le \frac{1}{3}e^{-64}$$

$a=5$ $\qquad$ $a=4.$

(c) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ compare with $\displaystyle\int_2^{\infty} \frac{dx}{x(\ln x)^2}$

$\frac{1}{x(\ln x)^2}$ positive, continuous and decreasing on $[2,\infty)$

$$\int_2^{\infty} \frac{dx}{x(\ln x)^2} = \lim_{N\to\infty} \int_2^{N} \frac{dx}{x(\ln x)^2} \quad \left|\begin{array}{l} u = \ln x \\ du = \frac{dx}{x} \\ x = 2 \longrightarrow u = \ln 2 \\ x = N \longrightarrow u = \ln N \end{array}\right|$$

$$= \lim_{N\to\infty} \int_{\ln 2}^{\ln N} \frac{du}{u^2} = \lim_{N\to\infty} \left(-\frac{1}{u}\right)_{\ln 2}^{\ln N} = -\lim_{N\to\infty} \cancel{\frac{1}{\ln N}}^{\,0} + \frac{1}{\ln 2} = \frac{1}{\ln 2}$$

$\lim_{N\to\infty} \ln N = \infty$

$\displaystyle\int_2^{\infty} \frac{dx}{x(\ln x)^2}$ converges and so is $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$.

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2. Explain why the Integral Test cannot be used for the series $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(\pi n)}{n^2+1}$.

$a_n = \frac{\cos(\pi n)}{n^2+1} = \frac{(-1)^n}{n^2+1}$, $n=1 \longrightarrow a_1 = -\frac{1}{1^2+1} = -\frac{1}{2}$

not positive the Integral Test is violated.

$a_2 = \frac{1}{5}$, $a_3 = -\frac{1}{10}$

$$\int_1^\infty \frac{dx}{x^p} - \begin{cases} \text{convergent, if } p>1 \\ \text{divergent, if } p\le 1. \end{cases}$$

3. The tenth partial sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is $s_{10} \approx 1.64522$.

   (a) Find the error when using the tenth partial sum to approximate the sum of the series.

   (b) How many terms n would be required so that the error $s \approx s_n$ is less than 0.001?

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the integral test. $\int_1^\infty \frac{dx}{x^2}$ $(p=2>1)$ convergent.

$$\int_a^\infty \frac{dx}{x^2} = \left(-\frac{1}{x}\right)_a^\infty = -\frac{1}{\infty} + \frac{1}{a} = \frac{1}{a}$$

$s \approx s_{10}$

$s = s_{10} + R_{10}$

$$\int_{11}^\infty \frac{dx}{x^2} \le R_{10} \le \int_{10}^\infty \frac{dx}{x^2}$$

$$\boxed{\frac{1}{11} \le R_{10} \le \frac{1}{10}}$$

(b) How many terms n would be required so that the error $s \approx s_n$ is less than 0.001?

$R_n \le 10^{-3}$

$$R_n = \int_n^\infty \frac{dx}{x^2} = \frac{1}{n} \le \frac{1}{10^3} \longrightarrow n \ge 10^3$$

$\cos^2 x = \frac{1+\cos 2x}{2}$, $\sin^2 x = \frac{1-\cos 2x}{2}$, $\sin 2x = 2\sin x \cos x$

$\tan^2 x = \sec^2 x - 1$

$\sin^2 x + \cos^2 x = 1$

4. Evaluate the integral

(a) $\int \sin^3 x \cos^4 x \, dx = \int \sin x \cdot \sin^2 x \cos^4 x \, dx$ $\quad \sin^2 x = 1-\cos^2 x$

$= \int \sin x (1-\cos^2 x)\cos^4 x \, dx$ $\quad u = \cos x,\ du = -\sin x \, dx$ $\quad = -\int (1-u^2)u^4 \, du$

$= -\int (u^4 - u^6)\, du = +\left(-\frac{u^5}{5} + \frac{u^7}{7}\right) + C = \boxed{\frac{\cos^7 x}{7} - \frac{\cos^5 x}{5} + C}$

(b) $\int \sin^2 x \cos^4 x \, dx = \int \sin^2 x (\cos^2 x)^2 dx = \int \frac{1-\cos 2x}{2}\left[\frac{1+\cos 2x}{2}\right]^2 dx$

$= \int \frac{1-\cos 2x}{2} \cdot \frac{1+2\cos 2x + \cos^2 2x}{4} dx = \frac{1}{8}\int (1-\cos 2x)(1+2\cos 2x + \cos^2 2x)\, dx$

$= \frac{1}{8}\int [1 + 2\cos 2x + \cos^2 2x - \cos 2x - 2\cos^2 2x - \cos^3 2x]\, dx$

$= \frac{1}{8}\int (1 + \cos 2x - \cos^2 2x - \cos^3 2x)\, dx = \frac{1}{8}\left[x + \cancel{\frac{1}{2}\sin 2x} - \frac{1}{2}x - \frac{1}{8}\sin 4x - \cancel{\frac{1}{2}\sin 2x} + \frac{1}{6}\sin^3 2x\right] + C$

$= \boxed{\frac{1}{8}\left[\frac{1}{2}x - \frac{1}{8}\sin 4x + \frac{1}{6}\sin^3 4x\right] + C}$

1 $\int \cos^2 2x \, dx = \int \frac{1+\cos 4x}{2} dx = \frac{1}{2}x + \frac{1}{8}\sin 4x + C$

2 $\int \cos^3 2x \, dx = \int \cos 2x (\cos^2 2x)\, dx = \int \cos 2x (1 - \sin^2 2x)\, dx$ $\quad u = \sin 2x,\ du = 2\cos 2x \, dx \rightarrow \cos 2x \, dx = \frac{du}{2}$

$= \frac{1}{2}\int (1-u^2)\, du = \frac{1}{2}\left(u - \frac{u^3}{3}\right) + C = \frac{1}{2}\left(\sin 2x - \frac{\sin^3 2x}{3}\right) + C$

(c) $\int_0^{\pi/4} \tan^4 x \sec^4 x \, dx = \int_0^{\pi/4} \tan^4 x \sec^2 x \cdot \sec^2 x \, dx$ $\quad \sec^2 x = \tan^2 x + 1$ $\quad = \int_0^{\pi/4} \tan^4 x (\tan^2 x + 1)\sec^2 x \, dx$

$u = \tan x$

$du = \sec^2 x \, dx$

$x = 0 \rightarrow u = \tan 0 = 0$

$x = \pi/4 \rightarrow u = \tan \frac{\pi}{4} = 1$

$= \int_0^1 (u^2+1)u^4 \, du = \int_0^1 (u^6 + u^4)\, du = \left(\frac{u^7}{7} + \frac{u^5}{5}\right)\Big|_0^1 = \frac{1}{7} + \frac{1}{5} = \boxed{\frac{12}{35}}$

(d) $\int \tan^3 x \sec^3 x \, dx = \int (\sec x \tan x)\tan^2 x \sec^2 x \, dx =$ $\quad u = \sec x,\ du = \sec x \tan x \, dx,\ \tan^2 x = \sec^2 x - 1 = u^2 - 1$

$= \int (u^2-1)u^2 \, du$

$= \int (u^4 - u^2)\, du$

$= \frac{u^5}{5} - \frac{u^3}{3} + C$

$= \boxed{\frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C}$

$\sqrt{x^2-a^2}$	$x = a\sec t$
$\sqrt{x^2+a^2}$	$x = a\tan t$
$\sqrt{a^2-x^2}$	$x = a\sin t$

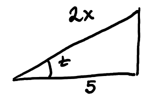
(e) $\int (4x^2-25)^{-3/2}\,dx$

$2x = 5\sec t$ or $x = \frac{5}{2}\sec t$

$4x^2 - 25 = 25\sec^2 t - 25 = 25\tan^2 t$

$dx = \frac{5}{2}\sec t\tan t\,dt$

$\frac{1}{\cos t} = \sec t = \frac{2x}{5} \rightarrow \cos t = \frac{5}{2x}$



$\sin t = \frac{\sqrt{4x^2-25}}{2x}$

$$= \int (25\tan^2 t)^{-3/2}\,\frac{5}{2}\sec t\tan t\,dt = \int \frac{\frac{5}{2}\sec t\tan t\,dt}{125\tan^3 t} = \frac{5}{2}\cdot\frac{1}{125}\int \frac{\sec t}{\tan^2 t}\,dt$$

$$= \frac{1}{50}\int \frac{\frac{1}{\cos t}}{\frac{\sin^2 t}{\cos^2 t}}\,dt = \frac{1}{50}\int \frac{\cos^2 t}{\sin^2 t\cdot\cos t}\,dt = \frac{1}{50}\int \frac{\cos t\,dt}{\sin^2 t}$$

$u = \sin t$, $du = \cos t\,dt$

$$= \frac{1}{50}\int \frac{du}{u^2} = \frac{1}{50}\left(-\frac{1}{u}\right) + C = -\frac{1}{50}\left(\frac{1}{\sin t}\right) + C = \boxed{-\frac{1}{50}\cdot\frac{2x}{\sqrt{4x^2-25}} + C}$$

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(f) $\int \frac{(x-1)^2\,dx}{5\sqrt{24-x^2+2x}}$

complete the square: $24 - x^2 + 2x = 24 - ([x^2 - 2x + 1] - 1)$

$= 24 - (x-1)^2 + 1$

$= 25 - (x-1)^2$

$$= \frac{1}{5}\int \frac{(x-1)^2\,dx}{\sqrt{25-(x-1)^2}}$$

$x - 1 = 5\sin t$ or $x = 1 + 5\sin t \longrightarrow 5\sin t = x - 1 \rightarrow \sin t = \frac{x-1}{5} \rightarrow t = \arcsin\left(\frac{x-1}{5}\right)$

$dx = 5\cos t\,dt$

$\sqrt{25-(x-1)^2} = \sqrt{25 - 25\sin^2 t} = 5\cos t \longrightarrow \frac{5\cos t}{5} = \frac{\sqrt{25-(x-1)^2}}{5}$

$$= \frac{1}{5}\int \frac{(5\sin t)^2\,\cancel{5\cos t}\,dt}{\cancel{5\cos t}} = 5\int \sin^2 t\,dt = 5\int \frac{1-\cos 2t}{2}\,dt = \frac{5}{2}\left(t - \frac{1}{2}\sin 2t\right) + C$$

$$= \frac{5}{2}t - \frac{5}{2}\cdot\frac{1}{2}\underbrace{2\sin t\cos t}_{\sin 2t} + C$$

$$= \boxed{\frac{5}{2}\cdot\arcsin\left(\frac{x-1}{5}\right) - \frac{5}{2}\cdot\frac{x-1}{5}\cdot\frac{\sqrt{25-(x-1)^2}}{5} + C}$$

(1) $\frac{1}{(x-a)} \to \frac{A}{x-a}$

(2) $\frac{1}{(x-a)^n} \to \frac{A}{x-a} + \frac{B}{(x-a)^2} + \dots + \frac{C}{(x-a)^n}$

(3) $\frac{1}{ax^2+bx+C} \to \frac{Ax+B}{ax^2+bx+C}$, $b^2-4ac<0$.

(g) $\int \frac{5x^2+x+12}{x^3+4x}dx$

$$\frac{5x^2+x+12}{x^3+4x} = \frac{5x^2+x+12}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

$$\frac{5x^2+x+12}{x(x^2+4)} = \frac{A(x^2+4)+(Bx+C)x}{x(x^2+4)}$$

$$5x^2+x+12 = Ax^2+4A+Bx^2+Cx$$

$$5x^2+x+12 = (A+B)x^2+Cx+4A$$

match up coefficients to the corresponding powers of x

x^2: $5=A+B \to B=5-A=5-3=2 \to \boxed{B=2}$

x: $\boxed{1=C}$

1: $4A=12 \to \boxed{A=3}$

$$\int \frac{5x^2+x+12}{x^3+4x}dx = \int \left[\frac{3}{x} + \frac{2x+1}{x^2+4} \right] dx$$

$$= 3\int \frac{dx}{x} + 2\int \frac{x\,dx}{x^2+4} + \int \frac{dx}{x^2+4}$$

$u = x^2+4$
$du = 2x\,dx$

$$= 3\ln|x| + \int \frac{du}{u} + \frac{1}{2}\arctan\frac{x}{2} + C$$

$$= 3\ln|x| + \ln|u| + \frac{1}{2}\arctan\frac{x}{2} + C$$

$$= \boxed{3\ln|x| + \ln|x^2+4| + \frac{1}{2}\arctan\frac{x}{2} + C}$$

$$\boxed{\int \frac{dx}{x^2+a^2} = \frac{1}{a}\arctan\frac{x}{a} + C}$$

$$\int \frac{dx}{x^2+1} = \arctan x + C$$

$u = \frac{x}{2}$, $du = \frac{dx}{2}$

$$\int \frac{dx}{x^2+4} = \int \frac{dx}{4\left(\frac{x^2}{4}+1\right)} = \frac{1}{4}\int \frac{dx}{\frac{x^2}{4}+1} = \frac{1}{2}\arctan\frac{x}{2} + C.$$

5. Determine whether the given integral is convergent or divergent.

(a) $\int_1^\infty \frac{4+\cos^4 x}{x}dx$

$0 \le \cos^4 x \le 1$

$4+0 \le 4+\cos^4 x \le 1+4$

$\frac{4}{x} \le \frac{4+\cos^4 x}{x} \le \frac{5}{x}$

$\int_1^\infty \frac{4+\cos^4 x}{x}dx$ is divergent by comparison with the integral $\int \frac{4}{x}dx$ $(p=1)$

(b) $\int_1^\infty \frac{3+\sin x}{x^2}dx$

$p=2>1$

$-1 \le \sin x \le 1$

$3-1 \le 3+\sin x \le 3+1$

$\frac{2}{x^2} \le \frac{3+\sin x}{x^2} \le \frac{4}{x^2}$

$\int_1^\infty \frac{3+\sin x}{x^2}dx$ converges by comparison with $\int_1^\infty \frac{4dx}{x^2}$ $(p=2>1)$

(c) $\int_0^\infty \frac{1}{\sqrt{x}+e^{4x}}dx$

$\frac{1}{\sqrt{x}+e^{4x}} \le \frac{1}{e^{4x}} = e^{-4x}$

leading term

$\int_0^\infty e^{-4x}dx = -\frac{1}{4}e^{-4x}\Big|_0^\infty = -\frac{1}{4}(\cancel{e^{-\infty}} - e^0) = \frac{1}{4} \rightarrow$ convergent

$\int_0^\infty \frac{dx}{\sqrt{x}+e^{4x}}$ converges by comparison with $\int_0^\infty e^{-4x}dx$.

6. Compute the following integrals or show that they diverge.

(a) $\int_e^\infty \frac{dx}{x \ln^5 x} = \lim_{N\to\infty} \int_e^N \frac{dx}{x \ln^5 x}$

$u = \ln x$, $du = \frac{dx}{x}$

$x = e \to u = \ln e = 1$

$x = N \to u = \ln N$

$= \lim_{N\to\infty} \int_1^{\ln N} \frac{du}{u^5} = \lim_{N\to\infty} \frac{u^{-5+1}}{-5+1}\Big|_1^{\ln N} = \lim_{N\to\infty} \left(\frac{u^{-4}}{-4}\right)\Big|_1^{\ln N}$

$(p = 5 > 1)$ – convergent.

$= -\frac{1}{4} \lim (\ln N)^{-4} + \frac{1}{4} = \frac{1}{4}$ – convergent.

(b) $\int_{-\infty}^0 (1+x)e^x \, dx = \lim_{N\to-\infty} \int_N^0 (1+x)e^x \, dx$

By parts:

D	I
+ $1+x$	e^x
− 1	e^x
0	e^x

$= \lim_{N\to-\infty} \left[(1+x)e^x - e^x\right]_N^0$

$= \lim_{N\to-\infty} \left(\cancel{e^0} - \cancel{e^0} - (1+N)e^N + \cancel{e^N}\right)$

$= -\lim_{N\to-\infty} (1+N)e^N \quad |\infty \cdot 0|$

$= -\lim_{N\to-\infty} \frac{1+N}{e^{-N}} \quad \left|\frac{\infty}{\infty}\right|$

L'Hospital's Rule

$-\lim_{N\to-\infty} \frac{1}{-e^{-N}} = \lim_{N\to-\infty} e^{-N} = \boxed{0}$ convergent.

(d) $\int_0^{2025} \frac{1}{\sqrt{2025-x}} dx$ $\left(\begin{array}{l} p = 1/2 \\ \text{convergent} \end{array}\right) = \lim_{b\to 2025^-} \int_0^b \frac{dx}{\sqrt{2025-x}} =$

$u = 2025 - x$, $du = -dx$

$x = 0 \to u = 2025$

$x = b \to u = 2025 - b$

$\boxed{\int_0^1 \frac{dx}{x^p} - \begin{cases} p < 1 & \text{convergent} \\ p \geq 1 & \text{divergent} \end{cases}}$

$= \lim_{b\to 2025^-} \int_{2025}^{2025-b} \frac{-du}{\sqrt{u}}$

$= -\lim_{b\to 2025^-} \frac{u^{1/2}}{1/2}\Big|_{2025}^{2025-b} = -2\left(\lim_{b\to 2025^-} \sqrt{2025-b} - \sqrt{2025}\right)$

$= \boxed{2\sqrt{2025}}$

(c) $\int_{-\infty}^\infty \frac{6x^5}{(x^6+3)^3} dx = \int_{-\infty}^0 + \int_0^\infty$

$p = 3 > 1$ – the integral is convergent.

11. Which of the following series is convergent?

(a) $\sum_{n=1}^{\infty} \frac{n^2}{n^{5/7}+1}$ $\leftarrow$ $a_n = \frac{n^2}{n^{5/7}+1}$, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{n^{5/7}+1} = \infty$

divergent by the Divergence Test.

(b) $\sum_{n=1}^{\infty} \frac{\pi^n}{3^n} = \sum_{n=1}^{\infty} \left(\frac{\pi}{3}\right)^n$ - geometric series,

$q = \frac{\pi}{3} > 1$, divergent.