MATH 150 - WEEK-IN-REVIEW 10

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PROBLEM STATEMENTS, SECTIONS 7.6, 8.1, 8.2 AND 8.3

This table is just a few of them!

FUNDAMENTAL TRIGONOMETRIC IDENTITIES

Reciprocal Identities

$$\csc(\theta) = \frac{1}{\sin(\theta)} \qquad \sec(\theta) = \frac{1}{\cos(\theta)} \qquad \cot(\theta) = \frac{1}{\tan(\theta)}$$

Quotient Identities

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} \qquad \cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$$

Pythagorean Identities

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

$$1 + \cot^2(\theta) = \csc^2\theta$$

$$\tan^2(\theta) + 1 = \sec^2\theta$$

Cofunctions

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta \qquad \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$

$$\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta \qquad \cot\left(\frac{\pi}{2} - \theta\right) = \tan\theta$$

$$\sec\left(\frac{\pi}{2} - \theta\right) = \csc\theta \qquad \csc\left(\frac{\pi}{2} - \theta\right) = \sec\theta$$

Even/Odd Identities

$$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta \qquad \tan(-\theta) = -\tan\theta$$

$$\csc(-\theta) = -\csc\theta \qquad \sec(-\theta) = \sec\theta \qquad \cot(-\theta) = -\cot\theta$$

Sum and Difference Identities

$$\sin(u+v) = \sin(u)\cos(v) + \cos(u)\sin(v)$$

$$\cos(u+v) = \cos(u)\cos(v) - \sin(u)\sin(v)$$

$$\tan(u+v) = \frac{\tan(u) + \tan(v)}{1 - \tan(u)\tan(v)}$$

$$\sin(u-v) = \sin(u)\cos(v) - \cos(u)\sin(v)$$

$$\cos(u-v) = \cos(u)\cos(v) + \sin(u)\sin(v)$$

$$\tan(u-v) = \frac{\tan(u) - \tan(v)}{1 + \tan(u)\tan(v)}$$

Double Angle Identities

$$\sin(2u) = 2\sin(u)\cos(u)$$

$$\begin{aligned}\cos(2u) &= \cos^2(u) - \sin^2(u)\\ &= 2\cos^2(u) - 1\\ &= 1 - 2\sin^2(u)\end{aligned}$$

1. Given $\cos(\theta) = \frac{4}{5}$ and $\underbrace{\csc(\theta) < 0}_{\sin(\theta)<0}$, find the value of $\cos(2\theta)$, $\sin(2\theta)$ and $\tan(2\theta)$ using the identities.

$\Rightarrow$ Quadrant IV

$\sin^2\theta + \cos^2\theta = 1$

$\sin^2\theta = 1 - \left(\frac{4}{5}\right)^2 = 1 - \frac{16}{25} = \frac{9}{25}$

$\sin\theta = -\frac{3}{5}$

$\sin(2\theta) = 2\sin\theta\cos(\theta) = 2\left(-\frac{3}{5}\right)\left(\frac{4}{5}\right) = \boxed{\frac{-24}{25}}$

$\cos(2\theta) = 2\cos^2\theta - 1 = 2\left(\frac{4}{5}\right)^2 - 1 = 2\left(\frac{16}{25}\right) - 1 = \boxed{\frac{7}{25}}$

$\tan(2\theta) = \frac{\sin(2\theta)}{\cos(2\theta)} = \frac{-\frac{24}{25}}{\frac{7}{25}} = \boxed{-\frac{24}{7}}$

2. State the domain and range of $y = \text{arcsec}(x)$

Domain: $(-\infty, -1] \cup [1, \infty)$

Range: $[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$

3. State the domain and range of $y = \arccos(x)$

Domain: $[-1, 1]$

Range: $[0, \pi]$

4. Simplify the expression $\dfrac{2\tan(7^\circ)}{1-\tan^2(7^\circ)}$ $= \tan(2(7^\circ)) = \tan(14^\circ)$

5. Find the exact value of $2\cos^2(22.5^\circ) - 1$ $= \cos(2 \times 22.5^\circ) = \cos(45^\circ)$

$= \dfrac{\sqrt{2}}{2}$

6. Write the following expression as product of trig functions.

$$\underset{\alpha}{\cos(9x)} + \underset{\beta}{\cos(2x)} = 2\cos\left(\frac{9x+2x}{2}\right)\cos\left(\frac{9x-2x}{2}\right)$$

$$= 2\cos\left(\frac{11x}{2}\right)\cos\left(\frac{7x}{2}\right)$$

used the sum to product ID

(no need to memorize)

7. Find the exact value of $\sin(\frac{5\pi}{12}) = \sin\left(\frac{2\pi}{3} - \frac{\pi}{4}\right)$

$\frac{5\pi}{12} = \frac{8\pi - 3\pi}{12} = \frac{2\pi}{3} - \frac{\pi}{4}$

*ID

$$= \sin\left(\frac{2\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right)\cos\left(\frac{2\pi}{3}\right)$$

$$= \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \times -\frac{1}{2}$$

$$= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} = \frac{\sqrt{6}+\sqrt{2}}{4}$$

8. Find the exact value of $\cos(\frac{19\pi}{8}) = \cos\left(\frac{\frac{19\pi}{4}}{2}\right) = -\sqrt{\frac{1+\cos(\frac{19\pi}{4})}{2}}$

$\frac{19\pi}{4} - 4\pi = \frac{3\pi}{4}$

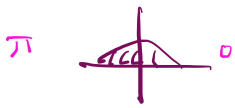
$$= -\sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}} = \sqrt{\frac{\frac{2-\sqrt{2}}{2}}{2}}$$

$$= \sqrt{\frac{2-\sqrt{2}}{4}} = \frac{\sqrt{2-\sqrt{2}}}{2}$$

I used half angle ID

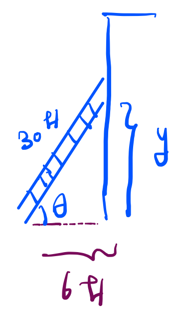
(No need to memorize)

9. Find the exact value of $\arccos(-\frac{1}{2})$ $= \frac{2\pi}{3}$



10. A 30-ft ladder is leaning against a building. If the base of the ladder is 6 ft from the base of the building, what is the angle of the elevation of the ladder? How high does the ladder reach on the building?

$\theta = ?$ $y = ?$



$\cos(\theta) = \frac{\text{adj.}}{\text{hyp.}} = \frac{6}{30} = \frac{1}{5}$

$\theta = \underbrace{\arccos\left(\frac{1}{5}\right)}_{\text{an acute angle}}$

$y^2 = (30)^2 - (6)^2$

$= 900 - 36$

$y^2 = 864$

$y = \sqrt{864}$ ft

11. Simplify each composition, if possible.

$\tan\left[\arctan(\frac{\sqrt{3}}{3})\right] =$ $\frac{\sqrt{3}}{3}$

in domain of $\arctan(\theta)$

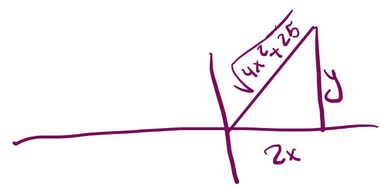
Note: $\frac{5\pi}{4}$ is not in restricted domain of sine $[-\frac{\pi}{2}, \frac{\pi}{2}]$

$\arcsin\left[\sin\left(\frac{5\pi}{4}\right)\right] =$ $\arcsin\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$

$\arccos\left[\tan(0)\right] =$ $\arccos(0) = \frac{\pi}{2}$

$\cot\left(\operatorname{arcsec}(1)\right) =$ $\cot(0)$ undefined

$$\tan\Big[\underbrace{\arccos\left(\frac{2x}{\sqrt{4x^2+25}}\right)}_{\theta}\Big] = \tan(\theta) = \frac{5}{2x}$$



$$\theta = \arccos\left(\frac{2x}{\sqrt{4x^2+25}}\right)$$

$$\cos\theta = \frac{2x}{\sqrt{4x^2+25}} = \frac{\text{adj.}}{\text{hyp.}}$$

$$y^2 + 4x^2 = \left(\sqrt{4x^2+25}\right)^2$$

$$y^2 + 4x^2 = 4x^2 + 25$$

$$y^2 = 25$$

$$y = 5$$

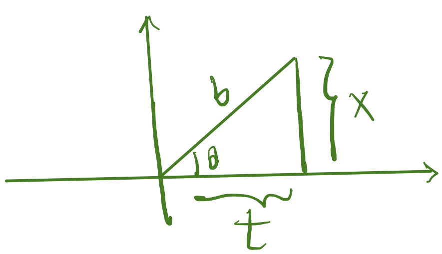
$$\sec\left[\arcsin\left(\frac{x}{b}\right)\right] = \sec(\theta) = \frac{b}{\sqrt{b^2 - x^2}}$$

$$\boxed{\text{for } -1 \leq \frac{x}{b} \leq 1}$$

$$-b \leq x \leq b$$

$$\sin\theta = \frac{a}{b}$$

$$\sec(\theta) = \frac{1}{\cos(\theta)} = \frac{\text{hyp.}}{\text{adj.}} = \frac{b}{\sqrt{b^2 - x^2}}$$

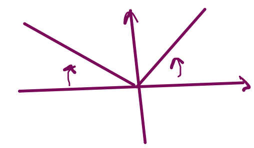


$$t^2 + a^2 = b^2$$

$$t = +\sqrt{b^2 - a^2}$$

12. Find all solutions to $2\sin(3\theta) - 1 = 0$, then state the solutions in the interval $[0, 2\pi)$.

$\sin(3\theta) = \frac{1}{2}$ let $u = 3\theta$ $\sin(u) = \frac{1}{2}$



$u_1 = \arcsin(\frac{1}{2}) + 2k\pi$ $u_2 = \pi - \arcsin(\frac{1}{2}) + 2k\pi$

$u = \frac{\pi}{6} + 2k\pi$ $u = \frac{5\pi}{6} + 2k\pi$

$3\theta = \frac{\pi}{6} + 2k\pi$ $3\theta = \frac{5\pi}{6} + 2k\pi$ $\frac{12k\pi}{18}$

$\boxed{\theta_1 = \frac{\pi}{18} + \frac{2k\pi}{3}}$ $\boxed{\theta_2 = \frac{5\pi}{18} + \frac{2k\pi}{3}}$

Solutions on $[0, 2\pi)$: $\underbrace{\frac{\pi}{18}, \frac{5\pi}{18}}_{k=0}, \underbrace{\frac{13\pi}{18}, \frac{17\pi}{18}}_{k=1}, \underbrace{\frac{25\pi}{18}, \frac{29\pi}{18}}_{k=2}$

$2\pi = \frac{36\pi}{18}$

13. Find all solutions to $\sin(2\theta - \pi)\cos(\theta) - 0.65\sin(2\theta - \pi) - \cos(\theta) + 0.65 = 0$, then state the solutions in the interval $[0, 2\pi)$.

factor by grouping: $\sin(2\theta - \pi) \cdot [\cos(\theta) - 0.65] - [\cos(\theta) - 0.65] = 0$

$(\sin(2\theta - \pi) - 1)(\cos(\theta) - 0.65) = 0$

$\sin(2\theta - \pi) - 1 = 0$ & $\cos(\theta) - 0.65 = 0$

$\sin(2\theta - \pi) = 1$ $\cos(\theta) = 0.65$

$2\theta - \pi = \arcsin(1) + 2k\pi$

$\boxed{\theta_2 = \arccos(0.65) + 2k\pi}$

$2\theta - \pi = \frac{\pi}{2} + 2k\pi$

$\boxed{\theta_3 = 2\pi - \arccos(0.65) + 2k\pi}$

$2\theta = \pi + \frac{\pi}{2} + 2k\pi$

$2\theta = \frac{3\pi}{2} + 2k\pi \Rightarrow \boxed{\theta_1 = \frac{3\pi}{4} + k\pi}$

on the interval $[0, 2\pi)$: $\underbrace{\frac{3\pi}{4}, \arccos(0.65), 2\pi - \arccos(0.65)}_{\text{let } k=0}, \underbrace{\frac{7\pi}{4}}_{k=1}$

14. Solve the equation $\sin(2x) + \cos(x) = 0$, then state the solutions in the interval $[0, 2\pi)$.

$2\sin(x)\cos(x) + \cos(x) = 0$

$\cos(x)(2\sin x + 1) = 0$

$\sin(x) = -\frac{1}{2}$

$\cos(x) = 0$

QIV $x = \arcsin\left(-\frac{1}{2}\right) + 2k\pi = -\frac{\pi}{6} + 2k\pi$

$x = \frac{\pi}{2} + k\pi$

QIII $x = \pi - \arcsin\left(-\frac{1}{2}\right) + 2k\pi = \frac{7\pi}{6} + 2k\pi$

on the interval $[0, 2\pi)$: $\underbrace{\frac{\pi}{2}, \frac{7\pi}{6}}_{k=0}, \underbrace{\frac{3\pi}{2}, \frac{11\pi}{6}}_{k=1}$,

15. Find all solutions to the equation $\dfrac{\cos(2x)}{\cos^2 x} = 1$. (Moved to next week)

16. Solve the equation $5\sin(\theta)\cot(\theta) + 4\cot(\theta) = 0$ (Moved to next week)

17. Find all solutions for $\tan(2x) + \tan x = 0$ on $[0, 2\pi)$ (Moved to next week)

18. Simplify $\tan(\arctan(3) - \arctan(2))$

α β

$$\tan(\alpha - \beta) = \frac{\tan(\alpha) - \tan(\beta)}{1 + \tan(\alpha)\cdot\tan(\beta)} = \frac{\tan(\arctan 3) - \tan(\arctan 2)}{1 + \tan(\arctan(3))\cdot\tan(\arctan(2))}$$

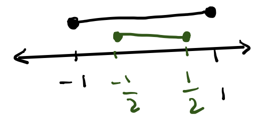
$$= \frac{3 - 2}{1 + (3)(2)} = \frac{1}{1+6} = \frac{1}{7}$$

19. Solve the trigonometric equation $\arcsin(x) = \arccos(2x)$

domain: $\arcsin(x)$: $-1 \le x \le 1$

$\arccos(2x)$: $-1 \le 2x \le 1$

$-\frac{1}{2} \le x \le \frac{1}{2}$

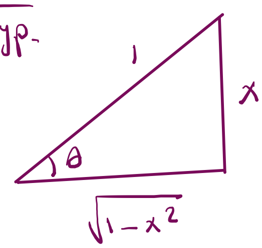


$x \in [-\frac{1}{2}, \frac{1}{2}]$

$$\cos(\arcsin(x)) = \cos(\arccos(2x))$$

$$\cos(\underbrace{\arcsin(x)}_{\theta}) = 2x$$

$\theta = \arcsin(x) \iff \sin(\theta) = \frac{x}{1} = \frac{\text{opp.}}{\text{hyp.}}$



$\cos(\theta) = \sqrt{1-x^2}$

$$\cos(\theta) = 2x$$

$$\sqrt{1-x^2} = 2x$$

$$1 - x^2 = 4x^2$$

$$1 = 5x^2$$

$$\frac{1}{5} = x^2 \qquad x = \pm\frac{1}{\sqrt{5}}$$

$x = +\frac{1}{\sqrt{5}} \approx \frac{1}{2.\ldots} < \frac{1}{2}$

in domain ✓

$x = -\frac{1}{\sqrt{5}} \approx -\frac{1}{2.\ldots} < -\frac{1}{2}$

Not in domain

Solution $x = \frac{1}{\sqrt{5}}$

20. Verify the following identities.

(a) $\sec^4(x) - \tan^4(x) = \sec^2(x) + \tan^2(x)$

$$LHS = \left(\sec^2(x) - \tan^2(x)\right)\left(\sec^2(x) + \tan^2(x)\right) = (1)\left(\sec^2(x) + \tan^2(x)\right)$$

$$= \sec^2(x) + \tan^2(x) = RHS$$

(b) $\dfrac{\cos(\theta)}{1-\sin(\theta)} = \dfrac{\sin(\theta)-\csc(\theta)}{\cos(\theta)-\cot(\theta)}$

$$RHS = \frac{\sin\theta - \frac{1}{\sin\theta}}{\cos\theta - \frac{\cos\theta}{\sin\theta}} = \frac{\frac{\sin^2\theta - 1}{\cancel{\sin\theta}}}{\frac{\sin\theta\cos\theta - \cos\theta}{\cancel{\sin\theta}}} = \frac{-\cos^2\theta}{\sin\theta\cos\theta - \cos\theta}$$

$$= \frac{\cos^{\cancel{2}}\theta}{\cancel{\cos\theta}(\sin\theta - 1)} = \frac{-\cos\theta}{\sin\theta - 1}$$

$$= \frac{\cos(\theta)}{1-\sin(\theta)} \quad RHS \checkmark$$

(c) $\dfrac{1-\cos(x)}{\sin(x)} + \dfrac{\sin(x)}{1-\cos(x)} = 2\csc(x)$

RHS take Common denom.

$$\frac{(1-\cos(x))^2 + \sin^2(x)}{\sin(x)(1-\cos(x))} =$$

$$\frac{1+\underbrace{\cos^2(x) - 2\cos(x) + \sin^2(x)}_{=1}}{\sin(x)(1-\cos(x))} = \frac{1-2\cos(x)+1}{\sin(x)(1-\cos(x))}$$

$$= \frac{2\cancel{(1-\cos(x))}}{\sin(x)\cancel{(1-\cos(x))}} = 2\csc(x)$$

✓ LHS

(d) $\dfrac{\sin(3x)+\cos(3x)}{\cos(x)-\sin(x)} = 1 + 4\sin(x)\cos(x)$

LHS

$$\frac{\sin(2x+x) + \cos(2x+x)}{\cos(x) - \sin(x)}$$

$$= \frac{\sin(2x)\cos(x) + \sin(x)\cos(2x) + \cos(2x)\cos(x) - \sin(2x)\sin(x)}{\cos(x) - \sin(x)}$$

factor by grouping

$$\frac{\sin(2x)(\cos(x)-\sin(x)) + \cos(2x)(\sin(x)+\cos(x))}{\cos(x) - \sin(x)}$$

$$= \sin(2x) + \frac{\cos(2x)\,(\sin(x)+\cos(x))}{\cos(x)-\sin(x)}$$

Conjugate of denom.

$$= \sin(2x) + \frac{\cos(2x)\,(\sin(x)+\cos(x))}{\cos(x)-\sin(x)} \times \frac{\cos(x)+\sin(x)}{\cos(x)+\sin(x)}$$

$$= \sin(2x) + \frac{\cos(2x)\cdot(\sin(x)+\cos(x))^2}{\cos^2(x)-\sin^2(x)}$$

$$= \sin(2x) + \frac{\cancel{\cos(2x)}\,(\sin(x)+\cos(x))^2}{\cancel{\cos(2x)}}$$

$$= \underbrace{\sin(2x)}_{=2\sin x\cos x} + \underbrace{\sin^2(x)+\cos^2(x)}_{=1} + 2\sin(x)\cos(x)$$

$$= 1 + 4\sin(x)\cos(x) \qquad \text{RHS} \checkmark$$