Math 140: Week-in-Review 9 (Chapters 5.3, 5.4)

1. Find the domain of each of the following functions, using interval notation. Then classify each domain restriction as the location of a vertical asymptote or hole in the graph of the function.

   (a) $f(x) = \dfrac{3x+7}{2x-5}$

   * $f(x)$ is a rational function
   $\Rightarrow$ denominator $\neq 0$

   $2x - 5 \neq 0$

   $\dfrac{2x}{2} \neq \dfrac{5}{2}$

   $x \neq 5/2 \Rightarrow$ Domain of $f(x)$:

   $\boxed{(-\infty, 5/2) \cup (5/2, \infty)}$

   * $x = 5/2$ is the location of a $\boxed{\text{vertical asymptote}}$ since $3x+7$ and $2x-5$ have no common terms and $2(5/2) - 5 = 0$.

   (b) $g(x) = \dfrac{4(x+4)}{(x+4)(x-3)}$

   * $f(x)$ is a rational function
   $\Rightarrow$ denom $\neq 0$

   $(x+4)(x-3) \neq 0$

   $x+4 \neq 0$ or $x-3 \neq 0$

   $x \neq -4$ or $x \neq 3$

   Domain of $g(x)$:

   $\boxed{(-\infty, -4) \cup (-4, 3) \cup (3, \infty)}$

   * $x = -4$ is the location of $\boxed{\text{a hole}}$ because $f(x) = \dfrac{4(x+4)}{(x+4)(x-3)}$ can be reduced to $r(x) = \dfrac{4}{x-3}$ with denominator $n(-4) = -4-3 \neq 0$

   * $x = 3$ is the location of $\boxed{\text{a vertical asymptote}}$ since $n(3) = 3-3 = 0$

(c) $h(x) = \dfrac{x(x+3)}{x^4 - 9x^2}$

* $h(x)$ is a rational function, denominator $\neq 0$

$$x^4 - 9x^2 = x^2(x^2-9)$$
$$= x^2(x+3)(x-3)$$
$$\neq 0$$

$x \neq 0$ or $x \neq -3$ or $x \neq 3$

Domain of $h(x)$:

$(-\infty, -3) \cup (-3, 0) \cup (0, 3) \cup (3, \infty)$

Domain restrictions: $x = 0, \pm 3$

$h(x) = \dfrac{x(x+3)}{x^2(x+3)(x-3)}$ reduces to $r(x) = \dfrac{1}{x(x-3)}$ after canceling terms

* $x = 0$ is the location of $\boxed{\text{a vertical asymptote}}$ since the denominator of $r(x)$ evaluates to $0(0-3) = 0$.
* $x = -3$ is the location of $\boxed{\text{a hole}}$ since the denominator of $r(x)$ evaluates to $-3(-3-3) = 18 \neq 0$.
* $x = 3$ is the location of $\boxed{\text{a vertical asymptote}}$ since the denominator of $r(x)$ evaluates to $3(3-3) = 0$.

2. Determine the x and y intercepts of the following functions.

(a) $f(x) = \dfrac{(3x+5)(x-2)}{7(x+4)}$

Domain: denom $\neq 0$, $7(x+4) \neq 0$

$(-\infty, -4) \cup (-4, \infty)$ $x \neq -4$

y-int: $\boxed{\left(0, -\frac{5}{14}\right)}$

$f(0) = \dfrac{(3\cdot 0+5)(0-2)}{7(0+4)} = \dfrac{(5)(-2)}{7(4)} = \dfrac{-10}{28} = \dfrac{-5}{14}$

x-int: $(3x+5)(x-2) = 0$

$3x+5=0 \Rightarrow \dfrac{3x}{3} = \dfrac{-5}{3} \Rightarrow x = -5/3$

$x-2=0 \Rightarrow x=2$

$\boxed{(-5/3, 0)}$ and $\boxed{(2, 0)}$

x-int(s)

(b) $g(x) = \dfrac{2x^3+3x^2-2x}{x(x+2)}$

Domain: denom $\neq 0$, $x(x+2) \neq 0$

$x \neq 0$ and $x \neq -2$

$(-\infty, -2) \cup (-2, 0) \cup (0, \infty)$

y-int: NONE because $g(0)$ DNE since $x=0$ is not in domain!

x-int(s): $2x^3+3x^2-2x = x(2x^2+3x-2) = 0$

$\Rightarrow x=0$ or $2x^2+3x-2=0$

not in domain!

$2x^2+4x-x-2=0$

$2x(x+2) - 1\cdot(x+2) = 0$

$(x+2)(2x-1) = 0$

$x=-2$ or $2x-1=0$

not in domain!

$\dfrac{2x}{2} = \dfrac{1}{2}$

$x = \dfrac{1}{2}$ ✓

$\boxed{\left(\frac{1}{2}, 0\right)}$

3. Compute and simplify the following as a single rational expression.

(a) $\left(\dfrac{x-5}{x+3}\right)+4\left(\dfrac{x+2}{x+3}\right)$ $= \dfrac{x-5}{x+3} + \dfrac{4(x+2)}{x+3}$

$$= \frac{x-5+4x+8}{x+3}$$

$$= \boxed{\frac{5x+3}{x+3}}$$

(b) $\dfrac{2x-3}{x-2}+\dfrac{x+3}{x+1}$ $= \dfrac{(x+1)}{(x+1)}\dfrac{2x-3}{x-2} + \dfrac{x+3}{x+1}\dfrac{(x-2)}{(x-2)}$

$$= \frac{(x+1)(2x-3)+(x+3)(x-2)}{(x+1)(x-2)}$$

$$= \frac{2x^2+\cancel{2x}-\cancel{3x}-3+x^2+\cancel{3x}-\cancel{2x}-6}{(x+1)(x-2)}$$

$$= \boxed{\frac{3x^2-9}{(x+1)(x-2)}}$$

(c) $\dfrac{\left(\dfrac{x+1}{x}\right)}{\left(\dfrac{2x+1}{x-1}\right)}$

(c) $\dfrac{x+1}{x}\cdot\dfrac{x-1}{2x+1} = \dfrac{(x+1)(x-1)}{x(2x+1)} = \boxed{\dfrac{x^2-1}{x(2x+1)}}$

4. For each of the following compute and completely simplify the difference quotient.

   (a) $f(x) = -2x^2 + 3x - 5$

$$\frac{f(x+h)-f(x)}{h} = \frac{\left(-2(x+h)^2+3(x+h)-5\right)-\left(-2x^2+3x-5\right)}{h}$$

$$= \frac{-2(x^2+2xh+h^2)+3(x+h)-5+2x^2-3x+5}{h}$$

$$= \frac{-\cancel{2x^2}-4xh-2h^2+\cancel{3x}+3h\cancel{-5}+\cancel{2x^2}\cancel{-3x}\cancel{+5}}{h}$$

$$= \frac{-4xh-2h^2+3h}{h}$$

$$= \frac{\cancel{h}}{\cancel{h}}(-4x-2h+3)$$

$$= \boxed{-4x-2h+3}$$

(b) $g(x) = \dfrac{x+4}{x-3}$

$$\frac{g(x+h)-g(x)}{h} = \frac{\dfrac{(x+h)+4}{(x+h)-3} - \dfrac{x+4}{x-3}}{h}$$

$$= \frac{1}{h}\left[\frac{(x-3)}{(x-3)}\,\frac{x+h+4}{x+h-3} - \frac{x+4}{x-3}\,\frac{(x+h-3)}{(x+h-3)}\right]$$

$$= \frac{1}{h}\left[\frac{(x-3)(x+h+4) - (x+4)(x+h-3)}{(x-3)(x+h-3)}\right]$$

$$= \frac{1}{h}\left[\frac{(\cancel{x^2}+\cancel{xh}+\cancel{4x}-\cancel{3x}-3h-\cancel{12})-(\cancel{x^2}+\cancel{xh}-\cancel{3x}+\cancel{4x}+4h-\cancel{12})}{(x-3)(x+h-3)}\right]$$

$$= \frac{1}{\cancel{h}}\left[\frac{-7\cancel{h}}{(x-3)(x+h-3)}\right]$$

$$= \boxed{\frac{-7}{(x-3)(x+h-3)}}$$

(c) $h(x) = \sqrt{3-2x}$

$$\frac{h(x+h)-h(x)}{h} = \frac{\sqrt{3-2(x+h)}-\sqrt{3-2x}}{h}$$

$$= \frac{\sqrt{3-2x-2h}-\sqrt{3-2x}}{h}$$

multiply & divide by the conjugate

$$= \frac{\sqrt{3-2x-2h}-\sqrt{3-2x}}{h} \cdot \frac{\left(\sqrt{3-2x-2h}+\sqrt{3-2x}\right)}{\left(\sqrt{3-2x-2h}+\sqrt{3-2x}\right)}$$

$$= \frac{(\cancel{3}-\cancel{2x}-2h)-(\cancel{3}-\cancel{2x})}{h\left(\sqrt{3-2x-2h}+\sqrt{3-2x}\right)}$$

$$= \frac{-2\cancel{h}}{\cancel{h}\left(\sqrt{3-2x-2h}+\sqrt{3-2x}\right)}$$

$$= \boxed{\frac{-2}{\sqrt{3-2x-2h}+\sqrt{3-2x}}}$$

5. Classify each of the following functions as a polynomial function (state its degree and leading coefficient), power function, rational function, or none of these (and explain why not).

   (a) $f(x) = \pi + x^3 - 4x^4$ $\boxed{\text{* polynomial function}}$

   * degree: 4
   * leading coefficient: -4

   (b) $h(z) = \dfrac{5z^2 - 12z}{2 + 3z^4}$ $\boxed{\text{* rational function}}$

   $= \dfrac{\text{polynomial}}{\text{polynomial}}$

   (c) $g(r) = \sqrt{6r^7}$ $\boxed{\text{* power function}}$

   $= \sqrt{6} \cdot r^{7/2}$

   (d) $m(x) = 1 + 3x^2 + 5x^{2/3} - 9x^7$ $\boxed{\text{* none of these}}$

   * not a polynomial because $5x^{2/3}$ has a non-integer power
   * not a power function because $m(x)$ has more than one term

   (e) $k(w) = \dfrac{w\sqrt[7]{w}}{w^2 - 4}$ $\boxed{\text{* none of these}}$

   * $k(w) = \dfrac{w^1 \cdot w^{1/7}}{w^2 - 4} = \dfrac{w^{8/7}}{w^2 - 4}$

   not a rational function since $w^{8/7}$ is not a polynomial

   not a power function since $k(w)$ can't be expressed as w^r, r a real number

   (f) $f(x) = \left(\dfrac{1}{3}\right)^x$ $\boxed{\text{* none of these}}$

   $f(x) = 3^{-x}$

   * $f(x)$ is not a power fn since the exponent (power) is variable (or polynomial)

6. Rewrite each radical in its equivalent exponential (power) form, assuming x is in the domain of each function

   (a) $\sqrt[7]{3x^2+5x}$

   $= \boxed{(3x^2+5x)^{1/7}}$

   (b) $12\sqrt{5x^2+3x-7}$

   $= \boxed{12(5x^2+3x-7)^{1/2}}$

   (c) $\sqrt[4]{(4-5x)^5}$

   $= \left[(4-5x)^5\right]^{\frac{1}{4}} = \boxed{(4-5x)^{5/4}}$

   (d) $\left(\sqrt[3]{3x^2+5x-2}\right)^7$

   $= \left[(3x^2+5x-2)^{1/3}\right]^7$

   $= \boxed{(3x^2+5x-2)^{7/3}}$

7. Rewrite each exponent function in its equivalent radical form, assuming x is in the domain of each function.

(a) $(x^2 + 2x)^{5/9}$

$= (x^2+2x)^{5\cdot\frac{1}{9}}$

$= \boxed{\sqrt[9]{(x^2+2x)^5}}$ or $\boxed{\left(\sqrt[9]{x^2+2x}\right)^5}$

(b) $(5x + 10)^{4/5}$

$= (5x+10)^{4\cdot\frac{1}{5}}$

$= \boxed{\sqrt[5]{(5x+10)^4}}$ or $\boxed{\left(\sqrt[5]{5x+10}\right)^4}$

(c) $3(5x - 2)^{7/2}$

$= 3(5x-2)^{7\cdot\frac{1}{2}} = \boxed{3\left(\sqrt{(5x-2)^7}\right)}$

or $\boxed{3\left(\sqrt{5x-2}\right)^7}$

(d) $-5 \cdot (4x^2 + 1)^{-9/4}$

$= -5(4x^2+1)^{-9\cdot\frac{1}{4}} = -5\cdot\left(\sqrt[4]{4x^2+1}\right)^{-9}$

$= \boxed{\dfrac{-5}{\left(\sqrt[4]{4x^2+1}\right)^9}}$ or $\boxed{\dfrac{-5}{\sqrt[4]{(4x^2+1)^9}}}$

8. State the domain of each function. Write your answer using interval notation. Then determine the x and y intercepts if possible.

(a) $f(x) = \sqrt[4]{3x - 27}$ * even root function

Domain of $f(x)$: $\boxed{* \ [9, \infty)}$

$3x - 27 \geq 0$

$\frac{3x}{3} \geq \frac{27}{3} \Rightarrow x \geq 9$

* y-int : NONE (since $x=0$ is not in the domain)

* x-int(s): $\sqrt[4]{3x-27} = 0 \Rightarrow 3x - 27 = 0 \Rightarrow x = 9$

$\boxed{(9, 0)}$

(b) $g(x) = 2\sqrt[3]{x - 8}$ * odd root function

* $x - 8$ is any real number $\Rightarrow$ x is any real number

* Domain of $g(x)$: $\boxed{(-\infty, \infty)}$

* y-int: $\boxed{(0, -4)}$

$g(0) = 2\sqrt[3]{-8} = 2 \cdot (-2) = -4$

* x-int: $2\sqrt[3]{x-8} = 0 \Rightarrow x = 8$

$\boxed{8, 0}$

(c) $h(x) = 5(2x + 16)^{3/4}$

$= 5\left(\sqrt[4]{2x+16}\right)^3$ * even root function

* $2x + 16 \geq 0 \Rightarrow \frac{2x}{2} \geq \frac{-16}{2} \Rightarrow \boxed{x \geq -8}$

* Domain of $h(x)$: $\boxed{[-8, \infty)}$

* y-int:

$h(0) = 5 \cdot \left(\sqrt[4]{16}\right)^3 = 5 \cdot 2^3 = 40$

$\boxed{(0, 40)}$

x-int : $5\left(\sqrt[4]{2x+16}\right)^3 = 0$

$2x + 16 = 0$

$x = -8$

$\boxed{(-8, 0)}$

9. State the domain of each function using interval notation.

(a) $f(x) = (5x-6)^{-4/3}$

$$= \frac{1}{(5x-6)^{4/3}} = \frac{1}{\left(\sqrt[3]{5x-6}\right)^4}$$

* denominator $\neq 0$ AND $\left(\sqrt[3]{5x-6}\right)^4$ is defined

* $\left(\sqrt[3]{5x-6}\right)^4$ defined when $\sqrt[3]{5x-6}$ is defined $\Rightarrow$ $5x-6$ any real #
* odd root * $\Downarrow$ x any real #

* denominator $\neq 0 \Rightarrow 5x-6 \neq 0 \Rightarrow x \neq \frac{6}{5}$
* DOMAIN of $f(x)$: $\boxed{(-\infty, 6/5) \cup (6/5, \infty)}$

(b) $h(x) = \dfrac{\sqrt{x+3}}{6\sqrt[3]{x+3}}$

* $\sqrt{x+3}$ (even root) defined when $x+3 \geq 0 \Rightarrow x \geq -3$

* $6\sqrt[3]{x+3}$ (odd root) defined when $x+3$ is defined $\Rightarrow$ x any real #

* $h(x)$ defined when denominator $\neq 0$
i.e. $x+3 \neq 0 \Rightarrow x \neq -3$

* DOMAIN OF $h(x)$: $\boxed{(-3, \infty)}$

(c) $g(r) = \dfrac{5r}{\sqrt{r+3}-4}$

* $5r$ defined for any r (polynomial)
* $\sqrt{r+3}$ (even root) defined when $r+3 \geq 0 \Rightarrow r \geq -3$
* $g(r)$ defined if denominator $\neq 0 \Rightarrow \sqrt{r+3} - 4 \neq 0$

$$\sqrt{r+3} \neq 4$$
$$r+3 \neq 16$$
$$r \neq 13$$

* DOMAIN OF $g(r)$:

$\boxed{[-3, 13) \cup (13, \infty)}$

10. Rationalize each numerator or denominator, as appropriate, and simplify the expression.

(a) $\dfrac{x-9}{\sqrt{x}-3} \cdot \dfrac{\sqrt{x}+3}{\sqrt{x}+3}$

$$= \frac{(x-9)(\sqrt{x}+3)}{(\sqrt{x})^2 - 3^2}$$

$$= \frac{\cancel{(x-9)}(\sqrt{x}+3)}{\cancel{x-9}} = \boxed{\sqrt{x}+3}$$

(b) $\dfrac{5+\sqrt{x}}{x-1} \cdot \dfrac{5-\sqrt{x}}{5-\sqrt{x}}$

$$= \frac{5^2 - x}{(x-1)(5-\sqrt{x})} = \boxed{\frac{25-x}{(x-1)(5-\sqrt{x})}}$$

(c) $\dfrac{\sqrt{x+h+1}-\sqrt{x+1}}{h} \cdot \dfrac{(\sqrt{x+h+1}+\sqrt{x+1})}{(\sqrt{x+h+1}+\sqrt{x+1})}$

$$= \frac{(\cancel{x}+h+\cancel{1}) - (\cancel{x}+\cancel{1})}{h(\sqrt{x+h+1}+\sqrt{x+1})}$$

$$= \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h+1}+\sqrt{x+1})} = \boxed{\frac{1}{\sqrt{x+h+1}+\sqrt{x+1}}}$$