Math 308: Week-in-Review 8 (6.3 - 6.5)

6.1-6.6 Laplace Transform

Review

- Definition of the Laplace transform

$$\mathcal{L}\{f\} = \int_0^\infty e^{-st} f(t)\, dt$$

- General strategy for solving differential equations with the Laplace transform
  1. Laplace transform
  2. Solve for $Y(s)$
  3. Inverse transform

- Common Laplace transforms

$f(t)$	$F(s)$	defined for
1	$\dfrac{1}{s}$	$s > 0$
e^{at}	$\dfrac{1}{s-a}$	$s > a$
$t^n (n = 1, 2, \ldots)$	$\dfrac{n!}{s^{n+1}}$	$s > 0$
$\sin(bt)$	$\dfrac{b}{s^2+b^2}$	$s > 0$
$\cos(bt)$	$\dfrac{s}{s^2+b^2}$	$s > 0$
$e^{at}t^n (n = 1, 2, \ldots)$	$\dfrac{n!}{(s-a)^{n+1}}$	$s > a$
$e^{at}\sin(bt)$	$\dfrac{b}{(s-a)^2+b^2}$	$s > a$
$e^{at}\cos(bt)$	$\dfrac{s-a}{(s-a)^2+b^2}$	$s > a$
$u_c(t)(c \geq 0)$	$\dfrac{e^{-cs}}{s}$	$s > 0$
$\delta(t-c)(c \geq 0)$	e^{-cs}	–

- Shift theorems

$$\begin{aligned}
\mathcal{L}\{u_c(t)f(t-c)\} &= e^{-cs}F(s) \\
\mathcal{L}\{u_c(t)f(t)\} &= e^{-cs}\mathcal{L}\{f(t+c)\} \\
\mathcal{L}^{-1}\{e^{-cs}F(s)\} &= u_c(t)f(t-c) \\
\mathcal{L}^{-1}\{F(s-c)\} &= e^{ct}f(t)
\end{aligned}$$

1 6.2: Solving ODEs with Laplace Transforms

Review

- Laplace transform of derivatives

$$\mathcal{L}\{f'\} = sF(s) - f(0)$$
$$\mathcal{L}\{f''\} = s^2F(s) - sf(0) - f'(0)$$
$$\mathcal{L}\{f'''\} = s^3F(s) - s^2f(0) - sf'(0) - f''(0)$$

- How to solve differential equations with the Laplace transform
  - Apply the Laplace transform to each term of the ODE.
  - Substitute initial conditions and solve for $Y(s)$.
  - Compute the inverse Laplace transform to find $y(t)$.

1. **Use the Laplace transform to solve the initial value problem**

$$y'' - 3y' + 2y = 0, \quad y(0) = -2, \quad y'(0) = 1.$$

SOLUTION 1 *Let's solve this step-by-step using the Laplace transform method.*

First, apply the Laplace transform to both sides of the differential equation. Define $Y(s) = \mathcal{L}\{y(t)\}$*. Using the properties of the Laplace transform for derivatives:*

- $\mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - s(-2) - 1 = s^2Y(s) + 2s - 1$, - $\mathcal{L}\{y'\} = sY(s) - y(0) = sY(s) - (-2) = sY(s) + 2$, - $\mathcal{L}\{y\} = Y(s)$, - $\mathcal{L}\{0\} = 0$.

So the equation becomes:

$$s^2Y(s) + 2s - 1 - 3[sY(s) + 2] + 2Y(s) = 0.$$

Distribute and simplify:

$$s^2Y(s) + 2s - 1 - 3sY(s) - 6 + 2Y(s) = 0.$$

Combine like terms:

$$(s^2 - 3s + 2)Y(s) + (2s - 7) = 0.$$

Solve for $Y(s)$*:*

$$(s^2 - 3s + 2)Y(s) = -2s + 7,$$

$$Y(s) = \frac{-2s + 7}{s^2 - 3s + 2}.$$

Factor the denominator: $s^2 - 3s + 2 = (s - 1)(s - 2)$*, so:*

$$Y(s) = \frac{-2s + 7}{(s - 1)(s - 2)}.$$

Now, use partial fractions to decompose $Y(s)$*:*

$$\frac{-2s + 7}{(s - 1)(s - 2)} = \frac{a}{s - 1} + \frac{b}{s - 2}.$$

Multiply through by $(s - 1)(s - 2)$*:*

$$-2s + 7 = a(s - 2) + b(s - 1).$$

Expand and equate coefficients:

$$-2s + 7 = as - 2a + bs - b = (a + b)s + (-2a - b).$$

- *Coefficient of* s*:* $a + b = -2$, - *Constant:* $-2a - b = 7$.

Solve the system: - *From* $a+b=-2$, *express* $b=-2-a$. - *Substitute into* $-2a-b=7$:

$$-2a-(-2-a)=7,$$

$$-2a+2+a=7,$$

$$-a+2=7,$$

$$-a=5,$$

$$a=-5.$$

- *Then* $b=-2-(-5)=-2+5=3$.

Thus:

$$Y(s)=\frac{-5}{s-1}+\frac{3}{s-2}.$$

Take the inverse Laplace transform: - $\mathcal{L}^{-1}\left\{\frac{-5}{s-1}\right\}=-5e^{t}$, - $\mathcal{L}^{-1}\left\{\frac{3}{s-2}\right\}=3e^{2t}$.

So:

$$y(t)=-5e^{t}+3e^{2t}.$$

Verify the initial conditions: - $y(0)=-5e^{0}+3e^{0}=-5+3=-2$, *matches.* - $y'(t)=-5e^{t}+6e^{2t}$, *so* $y'(0)=-5+6=1$, *matches.*

The solution is:

$$\boxed{y(t)=-5e^{t}+3e^{2t}}$$

2. **Use the Laplace transform to solve the initial value problem**

$$y'' + 2y' + 5y = 0, \quad y(0) = 1, \quad y'(0) = -1.$$

SOLUTION 2 *Apply the Laplace transform to the equation, with* $Y(s) = \mathcal{L}\{y(t)\}$*:*
- $\mathcal{L}\{y''\} = s^2Y(s) - s(1) - (-1) = s^2Y(s) - s + 1$, - $\mathcal{L}\{y'\} = sY(s) - 1$, - $\mathcal{L}\{y\} = Y(s)$, - $\mathcal{L}\{0\} = 0$.
The transformed equation is:

$$(s^2Y(s) - s + 1) + 2(sY(s) - 1) + 5Y(s) = 0.$$

Simplify:

$$s^2Y(s) - s + 1 + 2sY(s) - 2 + 5Y(s) = 0,$$
$$(s^2 + 2s + 5)Y(s) - s - 1 = 0.$$

Solve for $Y(s)$*:*

$$(s^2 + 2s + 5)Y(s) = s + 1,$$
$$Y(s) = \frac{s+1}{s^2 + 2s + 5}.$$

Complete the square in the denominator:

$$s^2 + 2s + 5 = (s+1)^2 + 4.$$

So:

$$Y(s) = \frac{s+1}{(s+1)^2 + 4}.$$

This matches the form $\mathcal{L}\{e^{at}\cos(bt)\} = \frac{s-a}{(s-a)^2 + b^2}$*, with* $a = -1$*,* $b = 2$*:*

$$\frac{s-(-1)}{(s-(-1))^2 + 2^2} = \frac{s+1}{(s+1)^2 + 4},$$

$$\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2 + 4}\right\} = e^{-t}\cos(2t).$$

Thus:

$$y(t) = e^{-t}\cos(2t).$$

Verify: - $y(0) = e^0\cos(0) = 1 \cdot 1 = 1$, - $y'(t) = -e^{-t}\cos(2t) - 2e^{-t}\sin(2t)$*, so* $y'(0) = -1 - 2 \cdot 0 = -1$.
The solution is:

$$\boxed{y(t) = e^{-t}\cos(2t)}$$

3. **Use the Laplace transform to solve the initial value problem**

$$y'' + 6y' + 9y = 3e^{-t}, \quad y(0) = 1, \quad y'(0) = 0.$$

SOLUTION 3 *Take the Laplace transform:*
- $\mathcal{L}\{y''\} = s^2Y(s) - s(1) - 0 = s^2Y(s) - s$, - $\mathcal{L}\{y'\} = sY(s) - 1$, - $\mathcal{L}\{y\} = Y(s)$, - $\mathcal{L}\{3e^{-t}\} = 3 \cdot \dfrac{1}{s+1} = \dfrac{3}{s+1}$.

The equation becomes:

$$(s^2Y(s) - s) + 6(sY(s) - 1) + 9Y(s) = \frac{3}{s+1}.$$

Simplify:

$$s^2Y(s) - s + 6sY(s) - 6 + 9Y(s) = \frac{3}{s+1},$$

$$(s^2 + 6s + 9)Y(s) - s - 6 = \frac{3}{s+1}.$$

Factor: $s^2 + 6s + 9 = (s+3)^2$, *so:*

$$(s+3)^2Y(s) - s - 6 = \frac{3}{s+1}.$$

Solve:

$$(s+3)^2Y(s) = \frac{3}{s+1} + s + 6.$$

Combine the right-hand side:

$$s + 6 = \frac{(s+6)(s+1)}{s+1} = \frac{s^2 + 7s + 6}{s+1},$$

$$\frac{3}{s+1} + s + 6 = \frac{3 + s^2 + 7s + 6}{s+1} = \frac{s^2 + 7s + 9}{s+1}.$$

Thus:

$$Y(s) = \frac{s^2 + 7s + 9}{(s+1)(s+3)^2}.$$

Partial fractions:

$$\frac{s^2 + 7s + 9}{(s+1)(s+3)^2} = \frac{a}{s+1} + \frac{b}{s+3} + \frac{c}{(s+3)^2}.$$

Multiply through:

$$s^2 + 7s + 9 = a(s+3)^2 + b(s+1)(s+3) + c(s+1).$$

Expand and equate:

$$(a+b)s^2 + (6a + 4b + c)s + (9a + 3b + c) = s^2 + 7s + 9.$$

Solve: - $a+b=1$, - $6a+4b+c=7$, - $9a+3b+c=9$.

From $a+b=1$, $b=1-a$. *Substitute:* - $6a+4(1-a)+c=7 \Rightarrow 2a+4+c=7 \Rightarrow c=3-2a$, - $9a+3(1-a)+(3-2a)=9 \Rightarrow 6a+3+3-2a=9 \Rightarrow 4a+6=9 \Rightarrow a=\frac{3}{4}$, - $b=1-\frac{3}{4}=\frac{1}{4}$, - $c=3-2\cdot\frac{3}{4}=\frac{3}{2}$.

So:

$$Y(s)=\frac{3/4}{s+1}+\frac{1/4}{s+3}+\frac{3/2}{(s+3)^2}.$$

Inverse transform:

$$y(t)=\frac{3}{4}e^{-t}+\frac{1}{4}e^{-3t}+\frac{3}{2}te^{-3t}.$$

Combine:

$$y(t)=\frac{3}{4}e^{-t}+\left(\frac{3}{2}t+\frac{1}{4}\right)e^{-3t}.$$

Verify: - $y(0)=\frac{3}{4}+\frac{1}{4}=1$, - $y'(0)=-\frac{3}{4}+\left(-\frac{3}{4}+\frac{3}{2}\right)=0$.

The solution is:

$$\boxed{y(t)=\frac{3}{4}e^{-t}+\left(\frac{3}{2}t+\frac{1}{4}\right)e^{-3t}}$$

4. **Use the Laplace transform to solve the initial value problem**

$$y''' - y' = 0, \quad y(0) = 1, \quad y'(0) = 2, \quad y''(0) = -1.$$

SOLUTION 4 *Apply the Laplace transform:*
- $\mathcal{L}\{y'''\} = s^3Y(s) - s^2(1) - s(2) - (-1) = s^3Y(s) - s^2 - 2s + 1$, - $\mathcal{L}\{y'\} = sY(s) - 1$, - $\mathcal{L}\{0\} = 0$.
The equation becomes:

$$(s^3Y(s) - s^2 - 2s + 1) - (sY(s) - 1) = 0,$$
$$(s^3 - s)Y(s) - s^2 - 2s + 2 = 0.$$

Solve:

$$Y(s) = \frac{s^2 + 2s - 2}{s(s^2 - 1)}.$$

Partial fractions:

$$\frac{s^2 + 2s - 2}{s(s-1)(s+1)} = \frac{a}{s} + \frac{b}{s-1} + \frac{c}{s+1}.$$

Multiply through:

$$s^2 + 2s - 2 = a(s-1)(s+1) + bs(s+1) + cs(s-1).$$

Expand:

$$(a + b + c)s^2 + (b - c)s - a = s^2 + 2s - 2.$$

Equate: - $a + b + c = 1$, - $b - c = 2$, - $-a = -2 \Rightarrow a = 2$, - $2 + b + c = 1 \Rightarrow b + c = -1$, - $b = 2 + c$, $2 + c + c = -1 \Rightarrow c = -\frac{3}{2}$, - $b = 2 - \frac{3}{2} = \frac{1}{2}$.
So:

$$Y(s) = \frac{2}{s} + \frac{1/2}{s-1} - \frac{3/2}{s+1}.$$

Inverse transform:

$$y(t) = 2 + \frac{1}{2}e^t - \frac{3}{2}e^{-t}.$$

Verify: - $y(0) = 2 + \frac{1}{2} - \frac{3}{2} = 1$, - $y'(0) = \frac{1}{2} + \frac{3}{2} = 2$, - $y''(0) = \frac{1}{2} - \frac{3}{2} = -1$.
The solution is:

$$\boxed{y(t) = 2 + \frac{1}{2}e^t - \frac{3}{2}e^{-t}}$$

2 6.3: Step Functions

Review

- The unit step function $u_c(t)$ is defined by

$$u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$$

- It can be used to write discontinuous functions into a single equation.
- The Laplace transform of $u_c(t)$ is

$$\mathcal{L}\{u_c(t)\} = \frac{e^{-cs}}{s}, \quad s > 0$$

- Laplace transforms of shifts

$$\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}F(s)$$
$$\mathcal{L}\{u_c(t)f(t)\} = e^{-cs}\mathcal{L}\{f(t+c)\}$$

- Inverse Laplace transform of shifts

$$\mathcal{L}^{-1}\{e^{-cs}F(s)\} = u_c(t)f(t-c)$$

5. **Convert the following function to a piecewise function. Also, graph the function. Compute its Laplace transform.**

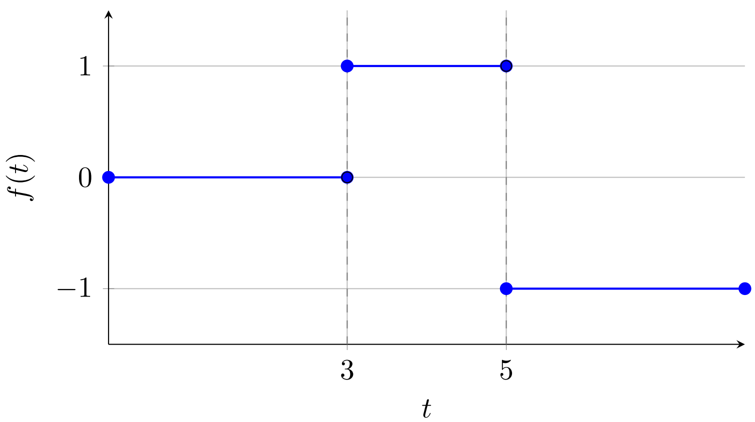
$$f(t) = u_3(t) - 2u_5(t)$$

SOLUTION 5 - *For* $t < 3$*:* $u_3(t) = 0$, $u_5(t) = 0$, *so* $f(t) = 0 - 2 \cdot 0 = 0$. - *For* $3 \leq t < 5$*:* $u_3(t) = 1$, $u_5(t) = 0$, *so* $f(t) = 1 - 2 \cdot 0 = 1$. - *For* $t \geq 5$*:* $u_3(t) = 1$, $u_5(t) = 1$, *so* $f(t) = 1 - 2 \cdot 1 = -1$.

Piecewise form:

$$f(t) = \begin{cases} 0 & t < 3 \\ 1 & 3 \leq t < 5 \\ -1 & t \geq 5 \end{cases}$$

Graph: The function is 0 until $t = 3$*, jumps to 1 from* $t = 3$ *to* $t < 5$*, then drops to* -1 *at* $t = 5$ *and remains there.*



Laplace transform:

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{u_3(t)\} - 2\mathcal{L}\{u_5(t)\} = \frac{e^{-3s}}{s} - 2\frac{e^{-5s}}{s} = \frac{e^{-3s} - 2e^{-5s}}{s}.$$

The Laplace transform is:

$$\boxed{\mathcal{L}\{f\} = \frac{e^{-3s} - 2e^{-5s}}{s}}$$

6. **Convert the following function to a piecewise function. Compute its Laplace transform.**

$$f(t) = t - \cos(t-2)u_2(t) - tu_3(t)$$

SOLUTION 6 - *For* $t < 2$*:* $u_2(t) = 0$*,* $u_3(t) = 0$*, so* $f(t) = t - 0 - 0 = t$*. - For* $2 \leq t < 3$*:* $u_2(t) = 1$*,* $u_3(t) = 0$*, so* $f(t) = t - \cos(t-2)$*. - For* $t \geq 3$*:* $u_2(t) = 1$*,* $u_3(t) = 1$*, so* $f(t) = t - \cos(t-2) - t = -\cos(t-2)$*.*

Piecewise form:

$$f(t) = \begin{cases} t & t < 2 \\ t - \cos(t-2) & 2 \leq t < 3 \\ -\cos(t-2) & t \geq 3 \end{cases}$$

Laplace transform:

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{t\} - \mathcal{L}\{\cos(t-2)u_2(t)\} - \mathcal{L}\{tu_3(t)\}.$$

- $\mathcal{L}\{t\} = \frac{1}{s^2}$, - $\mathcal{L}\{u_2(t)\cos(t-2)\} = e^{-2s}\frac{s}{s^2+1}$, - $tu_3(t) = u_3(t)(t-3) + 3u_3(t)$, - $\mathcal{L}\{u_3(t)(t-3)\} = e^{-3s}\frac{1}{s^2}$, - $\mathcal{L}\{3u_3(t)\} = 3\frac{e^{-3s}}{s}$, - $\mathcal{L}\{tu_3(t)\} = e^{-3s}\frac{1}{s^2} + 3\frac{e^{-3s}}{s} = e^{-3s}\frac{1+3s}{s^2}$.

So:

$$\mathcal{L}\{f\} = \frac{1}{s^2} - \frac{se^{-2s}}{s^2+1} - \frac{(1+3s)e^{-3s}}{s^2}.$$

The Laplace transform is:

$$\boxed{\mathcal{L}\{f\} = \frac{1}{s^2} - \frac{se^{-2s}}{s^2+1} - \frac{(1+3s)e^{-3s}}{s^2}}$$

7. **Convert the following piecewise function into a form that involves step functions.**

$$g(t) = \begin{cases} 0, & t < 2 \\ 3, & 2 \leq t < 5 \\ \sin(3t), & t \geq 5 \end{cases}$$

SOLUTION 7 *Express using step functions:* - $t < 2$: $g(t) = 0$, - $2 \leq t < 5$: $3(u_2(t) - u_5(t))$, - $t \geq 5$: $\sin(3t)u_5(t)$.

So:

$$g(t) = 3u_2(t) - 3u_5(t) + \sin(3t)u_5(t).$$

Check: - $t < 2$: $0 - 0 + 0 = 0$, - $2 \leq t < 5$: $3(1 - 0) + 0 = 3$, - $t \geq 5$: $3(1 - 1) + \sin(3t) = \sin(3t)$.

The step function form is:

$$\boxed{g(t) = 3u_2(t) - 3u_5(t) + \sin(3t)u_5(t)}$$

8. **Convert the following piecewise function into a form that involves step functions. Compute its Laplace transform.**

$$g(t) = \begin{cases} 3t, & t < 5 \\ e^{3t}, & t \geq 5 \end{cases}$$

SOLUTION 8 *Step function form:*

$$g(t) = 3t(1 - u_5(t)) + e^{3t}u_5(t).$$

Check: - $t < 5$: $3t \cdot 1 + 0 = 3t$, - $t \geq 5$: $3t \cdot 0 + e^{3t} = e^{3t}$.

Laplace transform:

$$\mathcal{L}\{g(t)\} = 3\mathcal{L}\{t\} - 3\mathcal{L}\{tu_5(t)\} + \mathcal{L}\{e^{3t}u_5(t)\}.$$

- $\mathcal{L}\{3t\} = \dfrac{3}{s^2}$, - $tu_5(t) = u_5(t)(t-5) + 5u_5(t)$, - $\mathcal{L}\{u_5(t)(t-5)\} = e^{-5s}\dfrac{1}{s^2}$, - $\mathcal{L}\{5u_5(t)\} = 5\dfrac{e^{-5s}}{s}$, - $3\mathcal{L}\{tu_5(t)\} = 3e^{-5s}\dfrac{1+5s}{s^2}$, - $\mathcal{L}\{e^{3t}u_5(t)\} = e^{15}e^{-5s}\dfrac{1}{s-3} = \dfrac{e^{15-5s}}{s-3}$.

So:

$$\mathcal{L}\{g\} = \frac{3}{s^2} - 3e^{-5s}\frac{1+5s}{s^2} + \frac{e^{15-5s}}{s-3}.$$

The Laplace transform is:

$$\boxed{\mathcal{L}\{g\} = \frac{3}{s^2} - 3e^{-5s}\left(\frac{1+5s}{s^2}\right) + \frac{e^{15-5s}}{s-3}}$$

9. **Solve the initial value problem.**

$$f'' + 4f = u_3(t), \quad f(0) = 0, \quad f'(0) = 0.$$

SOLUTION 9 *Laplace transform:* - $\mathcal{L}\{f''\} = s^2F(s)$, - $\mathcal{L}\{f\} = F(s)$, - $\mathcal{L}\{u_3(t)\} = \frac{e^{-3s}}{s}$.

$$s^2F(s) + 4F(s) = \frac{e^{-3s}}{s},$$

$$F(s) = \frac{e^{-3s}}{s(s^2+4)}.$$

Partial fractions:

$$\frac{1}{s(s^2+4)} = \frac{a}{s} + \frac{bs+c}{s^2+4},$$

$$1 = a(s^2+4) + (bs+c)s,$$

- $a = \frac{1}{4}$, - $b = -\frac{1}{4}$, - $c = 0$.

So:

$$F(s) = e^{-3s}\left(\frac{1/4}{s} - \frac{1/4s}{s^2+4}\right).$$

Inverse transform:

$$f(t) = \frac{1}{4}u_3(t)[1 - \cos(2(t-3))].$$

The solution is:

$$\boxed{f(t) = \frac{1}{4}u_3(t)[1 - \cos(2(t-3))]}$$

10. **Solve the initial value problem.**

$$w'' + 2w' = \begin{cases} 3, & t < 5 \\ 0, & t \geq 5 \end{cases}, \quad w(0) = 0, \quad w'(0) = 0.$$

SOLUTION 10 *Rewrite:* $w'' + 2w' = 3(1 - u_5(t))$.

Laplace transform:

$$s^2W(s) + 2sW(s) = 3\left(\frac{1}{s} - \frac{e^{-5s}}{s}\right),$$

$$W(s) = \frac{3(1 - e^{-5s})}{s^2(s+2)}.$$

Partial fractions:

$$\frac{3}{s^2(s+2)} = -\frac{3/4}{s} + \frac{3/2}{s^2} + \frac{3/4}{s+2},$$

$$W(s) = \left(-\frac{3/4}{s} + \frac{3/2}{s^2} + \frac{3/4}{s+2}\right)(1 - e^{-5s}).$$

Inverse transform:

$$\boxed{w(t) = \left(-\frac{3}{4} + \frac{3}{2}t + \frac{3}{4}e^{-2t}\right) - u_5(t)\left[-\frac{3}{4} + \frac{3}{2}(t-5) + \frac{3}{4}e^{-2(t-5)}\right]}$$

11. **Consider a spring and mass system with a 5 kg mass hanging on a spring. When the mass is hung on the spring, the spring extends 50 cm. The mass experiences a damping force of 8 N when the mass is moving 2 m/s. The mass starts from equilibrium at rest, but there is an external force** $\cos(t)$ **that lasts for the first** 3π **seconds. Write down the initial value problem that describes this situation.**

SOLUTION 11 - *Mass* $m = 5\,kg$, - $k = \dfrac{mg}{extension} = \dfrac{5 \cdot 10}{0.5} = 100\,N/m$, - *Damping:* $c \cdot 2 = 8 \Rightarrow c = 4\,Ns/m$, - *Force:* $\cos(t)(1 - u_{3\pi}(t))$, - *Initial conditions:* $u(0) = 0$, $u'(0) = 0$.

IVP:

$$\boxed{5u'' + 4u' + 100u = \cos(t)(1 - u_{3\pi}(t)), \quad u(0) = 0, \quad u'(0) = 0}$$

3 6.6: Delta Functions

Review

- The Dirac delta function $\delta(t-c)$ is defined by

$$\int_{-\infty}^{\infty} f(t)\delta(t-c)\,dt = f(c), \quad c \geq 0$$

- It models instantaneous impulses.
- The Laplace transform of $\delta(t-c)$ is

$$\mathcal{L}\{\delta(t-c)\} = e^{-cs}$$

- Laplace transforms involving delta functions

$$\mathcal{L}\{g(t)\delta(t-c)\} = g(c)e^{-cs}$$

12. **Find the Laplace transform of the following function:**

$$f(t) = t^2\delta(t-3) + e^t\delta(t-5).$$

SOLUTION 12 *Using the property:* - $\mathcal{L}\{t^2\delta(t-3)\} = (3)^2e^{-3s} = 9e^{-3s}$, - $\mathcal{L}\{e^t\delta(t-5)\} = e^5e^{-5s}$.

$$\mathcal{L}\{f\} = 9e^{-3s} + e^{5-5s}.$$

The Laplace transform is:

$$\boxed{\mathcal{L}\{f\} = 9e^{-3s} + e^{5-5s}}$$

13. **Solve the initial value problem:**

$$y'' + 2y' + y = \delta(t-3), \quad y(0) = 1, \quad y'(0) = -1.$$

SOLUTION 13 *Laplace transform:* - $\mathcal{L}\{y''\} = s^2Y(s) - s(1) - (-1) = s^2Y(s) - s + 1$, - $\mathcal{L}\{y'\} = sY(s) - 1$, - $\mathcal{L}\{y\} = Y(s)$, - $\mathcal{L}\{\delta(t-3)\} = e^{-3s}$.

$$(s^2Y(s) - s + 1) + 2(sY(s) - 1) + Y(s) = e^{-3s},$$
$$(s^2 + 2s + 1)Y(s) - s - 1 = e^{-3s},$$
$$(s+1)^2Y(s) = e^{-3s} + s + 1,$$
$$Y(s) = \frac{e^{-3s}}{(s+1)^2} + \frac{s+1}{(s+1)^2} = e^{-3s}\frac{1}{(s+1)^2} + \frac{1}{s+1}.$$

Inverse transform:

$$y(t) = u_3(t)(t-3)e^{-(t-3)} + e^{-t}.$$

The solution is:

$$\boxed{y(t) = e^{-t} + u_3(t)(t-3)e^{-(t-3)}}$$

14. **A 2 kg mass is suspended from a spring and damper. When the mass is hung at rest, it stretches the spring by 2 meters. When the mass moves at 1 m/s, the damper exerts a resistive force of 4 N. At $t = 2$ seconds, the system is struck with a hammer, delivering an instantaneous impulse force of magnitude 3 N. The mass starts motion from equilibrium with an initial upward velocity of 0.5 m/s.**

   (a) Determine the spring constant k and damping coefficient c.

   (b) Write the governing differential equation for the displacement $u(t)$.

   (c) Solve for $u(t)$ and describe the motion of the system.

   (Use $g = 10\,\mathrm{m/s^2}$.)

SOLUTION 14 *(a) Parameters: -* $k = \frac{mg}{extension} = \frac{2 \cdot 10}{2} = 10\,N/m$, *-* $c = \frac{4}{1} = 4\,Ns/m$.

(b) Equation: - Upward velocity $u'(0) = -0.5$ *(downward positive), -* $2u'' + 4u' + 10u = 3\delta(t-2)$, $u(0) = 0$, $u'(0) = -0.5$.

(c) Solve:

$$2(s^2U(s) + 0.5) + 4sU(s) + 10U(s) = 3e^{-2s},$$

$$(2s^2 + 4s + 10)U(s) = 3e^{-2s} - 1,$$

$$U(s) = \frac{3e^{-2s} - 1}{2(s^2 + 2s + 5)},$$

$$U(s) = -\frac{1}{2}\frac{1}{(s+1)^2 + 4} + \frac{3}{2}e^{-2s}\frac{1}{(s+1)^2 + 4}.$$

Inverse transform:

$$u(t) = -\frac{1}{4}e^{-t}\sin(2t) + \frac{3}{4}u_2(t)e^{-(t-2)}\sin(2(t-2)).$$

Motion: The mass oscillates with decaying amplitude; the impulse at $t = 2$ *increases the amplitude temporarily.*

Answers:

(a) $\boxed{k = 10\,N/m, \quad c = 4\,Ns/m}$

(b) $\boxed{2u'' + 4u' + 10u = 3\delta(t-2), \quad u(0) = 0, \quad u'(0) = -0.5}$

(c) $\boxed{u(t) = -\frac{1}{4}e^{-t}\sin(2t) + \frac{3}{4}u_2(t)e^{-(t-2)}\sin(2(t-2))}$

