Fubini's Theorem If f is continuous on the rectangle

$$R = \{(x, y) \mid a \le x \le b, c \le y \le d\},$$

then

$$\iint_R f(x, y)\, dA = \int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dx\, dy$$

Example 1 (15.1). *Calculate the double integrals.*

(a) $\displaystyle\iint_R \frac{y \cos x}{y^2 + 1}\, dA$, *where* $R = \{(x, y) | -\pi/2 \le x \le \pi/2,\ 0 \le y \le 2\}$

$$f(x,y) = \frac{y\cos x}{y^2+1} = \frac{y}{y^2+1} \cdot \cos x .$$

Let $u = y^2+1$

$\Rightarrow du = 2y\,dy$

$$\iint \frac{y\cos x}{y^2+1}\, dA = \left(\int_{-\pi/2}^{\pi/2} \cos x\, dx\right)\left(\int_0^2 \frac{y}{y^2+1}\, dy\right)$$

$$= \sin x \Big|_{x=-\pi/2}^{\pi/2} \cdot \frac{1}{2}\int \frac{1}{u}\, du$$

$$= 2 \cdot \frac{1}{2} \ln u \Big|$$

$$= \ln(y^2+1) \Big|_{y=0}^{2} = \ln 5$$

(b) $\int_0^{\pi/2} \int_{-1}^{1} (x + x^2 \sin y)\, dx\, dy$ $= \int_0^{\frac{\pi}{2}} \left[\frac{x^2}{2} + \frac{x^3}{3} \sin y \right]_{x=-1}^{1} dy$

$= \int_0^{\frac{\pi}{2}} \left[\cancel{\tfrac{1}{2}} + \tfrac{1}{3}\sin y - \left(\cancel{\tfrac{1}{2}} - \tfrac{1}{3}\sin y\right)\right] dy$

$= \frac{2}{3} \int_0^{\frac{\pi}{2}} \sin y \, dy$

$= -\frac{2}{3} \cos y \Big|_0^{\frac{\pi}{2}} = \frac{2}{3}$

(c) $\iint_R xe^{-xy}\, dA$, *where* $R = [0,1] \times [0,2]$.

$= \int_0^1 \int_0^2 x e^{-xy}\, dy\, dx = \int_0^1 x \cdot \left[\frac{e^{-xy}}{-x} \right]_{y=0}^{2} dx$

$= -\int_0^1 \left(e^{-2x} - 1\right) dx$

$= -\left[\frac{e^{-2x}}{-2} - x \right]_{x=0}^{1}$

$= -\left[\left(\frac{e^{-2}}{-2} - 1\right) + \frac{1}{2}\right] = \frac{1}{2} + \frac{e^{-2}}{2} = \frac{1+e^{-2}}{2}$

If f is continuous on a ***type I region*** *D described by*

$$D = \{(x, y) \mid a \le x \le b, g_1(x) \le y \le g_2(x)\}$$

then

$$\iint\limits_D f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

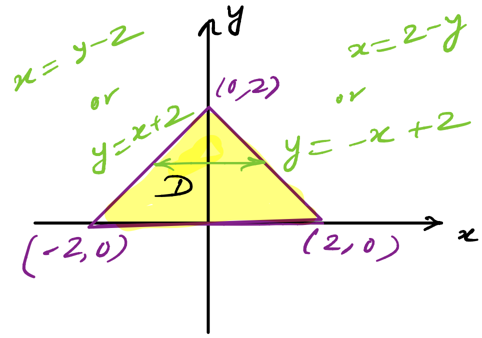
If f is continuous on a ***type II region*** *D described by*

$$D = \{(x, y) \mid c \le y \le d, h_1(y) \le x \le h_2(y)\}$$

then

$$\iint\limits_D f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy$$

Example 2 (15.2)**.** *Compute the double integral* $\iint\limits_D (2x + y^2)\, dA$*, where D is the triangular region with vertices $(-2, 0)$, $(0, 2)$ and $(2, 0)$.*



The region is both Type I and Type II. But considering the region as Type II minimizes the amount of work.

$D = \{ (x, y) : 0 \le y \le 2, \quad y-2 \le x \le 2-y \}$

$$\iint\limits_D (2x + y^2)\, dA = \int_0^2 \int_{y-2}^{2-y} (2x + y^2)\, dx\, dy = \int_0^2 \Big[x^2 + xy^2 \Big]_{x=y-2}^{2-y} dy$$

$$= \int_0^2 \Big[\cancel{(2-y)^2} + y^2(2-y) - \big(\cancel{(y-2)^2} + y^2(y-2) \big) \Big]\, dy$$

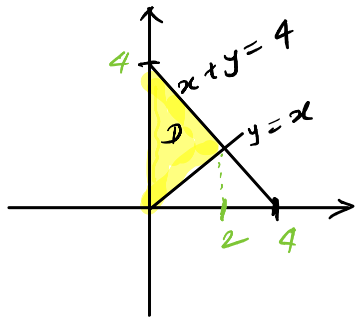
$$= \int_0^2 4y^2 - 2y^3\, dy = \frac{4y^3}{3} - \frac{2y^4}{4} \Big|_0^2$$

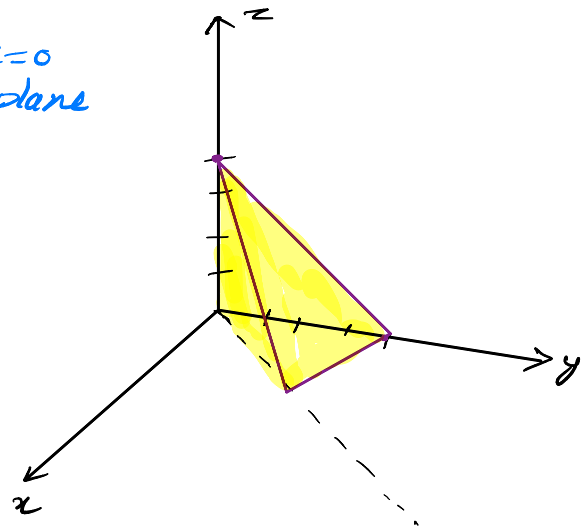
$$= \frac{32}{3} - 8 = \frac{8}{3}$$

Example 3 (15.2). *Find the volume of the tetrahedron bounded by the planes $z = 0$, $x = 0$, $x = y$ and $x + y + z = 4$.* $\Rightarrow z = 4 - x - y$

The intersection of $x+y+z=4$ and the xy-plane ($z=0$)
(or the projection of the tetrahedron) is

$x + y = 4$





So, $D = \{(x,y) : 0 \le x \le 2,\ x \le y \le 4-x\}$

$$\text{Volume} = \iint_D (4 - x - y)\, dA$$

$$= \int_0^2 \int_x^{4-x} (4 - x - y)\, dy\, dx$$

$$= \int_0^2 \left[4y - xy - \frac{y^2}{2} \right]_{y=x}^{4-x} dx$$

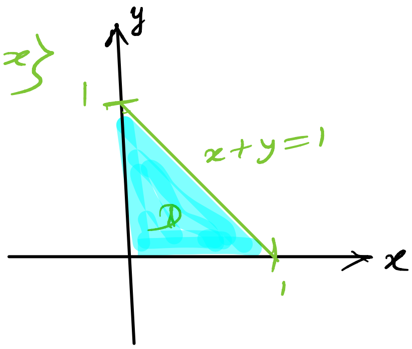
$$= \int_0^2 8 - 8x + 2x^2 \, dx$$

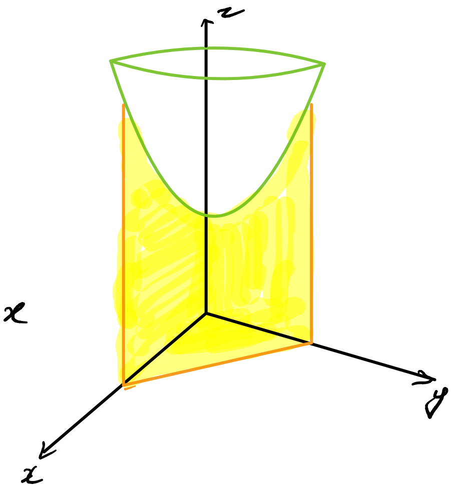
$$= 8x - 4x^2 + \frac{2x^3}{3} \Big|_0^2$$

$$= \frac{16}{3}$$

Example 4 (15.2). *Find the volume of the solid bounded by the paraboloid* $z = 1 + 3x^2 + 3y^2$ *and the planes* $x = 0$, $y = 0$, $z = 0$ *and* $x + y = 1$.

$D = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le 1-x\}$





$$\text{Volume} = \iint_D (1+3x^2+3y^2)\,dA$$

$$= \int_0^1 \int_0^{1-x} (1+3x^2+3y^2)\,dy\,dx$$

$$= \int_0^1 \Big[y + 3x^2y + y^3\Big]_{y=0}^{1-x}\,dx$$

$$= \int_0^1 \big[1-x+3x^2-3x^3+1-x^3+3x^2-3x\big]\,dx$$

$$= \int_0^1 2-4x+6x^2-4x^3\,dx$$

$$= 2x-2x^2+2x^3-x^4\Big|_0^1$$

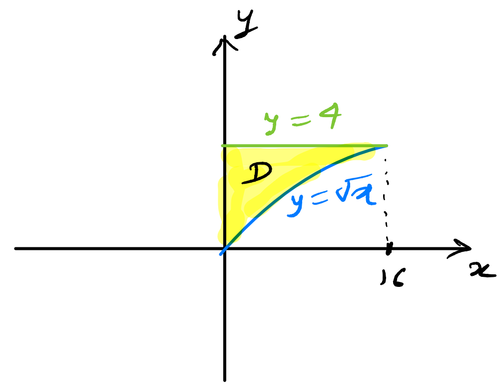
$$= 1$$

Example 5 (15.2)**.** *Sketch the region and change the order of integration.*

(a) $\displaystyle\int_0^{16}\int_{\sqrt{x}}^{4} f(x,y)\,dy\,dx$

Type I

$D = \{(x,y) : 0 \le x \le 16,\ \sqrt{x} \le y \le 4\}$



Type II : $D = \{(x,y) : 0 \le y \le 4,\ 0 \le x \le y^2\}$

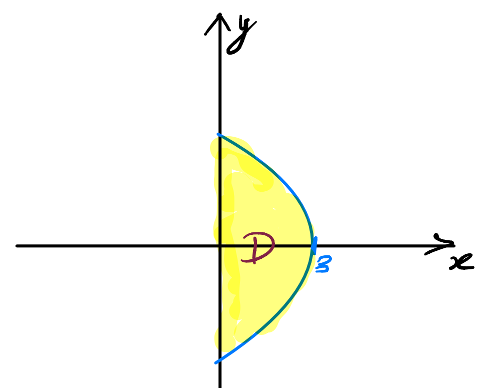
So,

$$\int_0^{16}\int_{\sqrt{x}}^{4} f(x,y)\,dy\,dx = \int_0^{4}\int_0^{y^2} f(x,y)\,dx\,dy$$

(b) $\displaystyle\int_{-3}^{3}\int_0^{\sqrt{9-y^2}} f(x,y)\,dx\,dy$

Type II

$D = \{(x,y) : -3 \le y \le 3,\ 0 \le x \le \sqrt{9-y^2}\}$



Note that $x = \sqrt{9-y^2} \Rightarrow x^2 = 9 - y^2$

$\Rightarrow x^2 + y^2 = 9.$

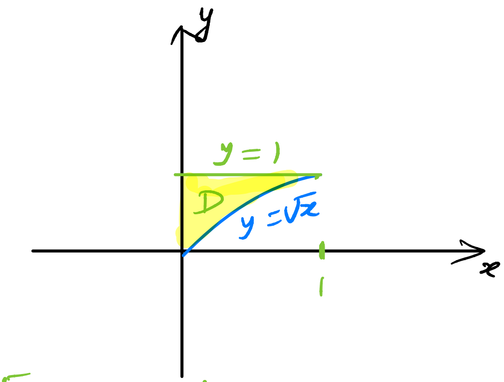
$\Rightarrow y = \pm\sqrt{9-x^2}$

So, Type II
$D = \{(x,y) : 0 \le x \le 3,\ -\sqrt{9-x^2} \le y \le \sqrt{9-x^2}\}.$

$$\int_{-3}^{3}\int_0^{\sqrt{9-y^2}} f(x,y)\,dx\,dy = \int_0^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} f(x,y)\,dx\,dy$$

Example 6 (15.2). *Evaluate the integral by reversing the order of integration.*

$$\int_0^1 \int_{\sqrt{x}}^1 x\,\sin(y^5+1)\,dy\,dx$$



$D = \{(x,y) : 0 \le x \le 1,\ \sqrt{x} \le y \le 1\}$

$= \{(x,y) : 0 \le y \le 1,\ 0 \le x \le y^2\}$

$$\int_0^1 \int_{\sqrt{x}}^1 x \sin(y^5+1)\,dy\,dx = \int_0^1 \int_0^{y^2} x \sin(y^5+1)\,dx\,dy$$

$$= \int_0^1 \sin(y^5+1) \cdot \frac{x^2}{2}\Big|_{x=0}^{x=y^2} dy$$

$$= \frac{1}{2}\int_0^1 y^4 \sin(y^5+1)\,dy$$

Let $u = y^5+1$

$du = 5y^4\,dy$

$$= \frac{1}{10}\int \sin u\,du$$

$$= -\frac{1}{10}\cos u\Big| = -\frac{1}{10}\cos(y^5+1)\Big|_{y=0}^{1}$$

$$= -\frac{1}{10}\left[\cos 2 - \cos 1\right]$$

If f is continuous on a polar rectangle R given by $0 \leq a \leq r \leq b$, $\alpha \leq \theta \leq \beta$, where $0 \leq \beta - \alpha \leq 2\pi$, then

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

If f is continuous on a polar region of the form

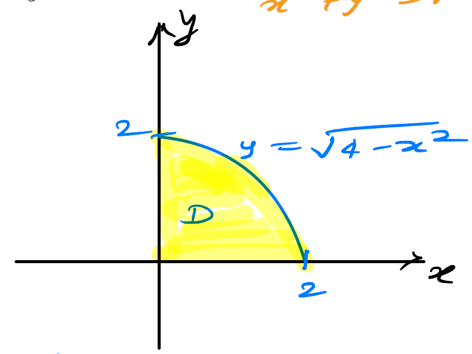
$$D = \{(r,\theta) \mid \alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta)\},$$

then

$$\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta) r\, dr\, d\theta$$

$x = r\cos\theta$
$y = r\sin\theta$
$x^2+y^2 = r^2$

Example 7 (15.3). *Evaluate the integral* $\displaystyle\int_0^2 \int_0^{\sqrt{4-x^2}} \sqrt{1+x^2+y^2}\, dy\, dx$



$D = \{(x,y) : 0 \leq x \leq 2,\ 0 \leq y \leq \sqrt{4-x^2}\}$

$y = \sqrt{4-x^2} \Rightarrow x^2+y^2 = 4.$

$\Rightarrow r^2 = 4 \Rightarrow r = 2$

So, $D = \{(r,\theta) : 0 \leq r \leq 2,\ 0 \leq \theta \leq \frac{\pi}{2}\}$

$$\int_0^2 \int_0^{\sqrt{4-x^2}} \sqrt{1+x^2+y^2}\, dy\, dx = \int_0^{\frac{\pi}{2}} \int_0^2 \sqrt{1+r^2}\; r\, dr\, d\theta$$

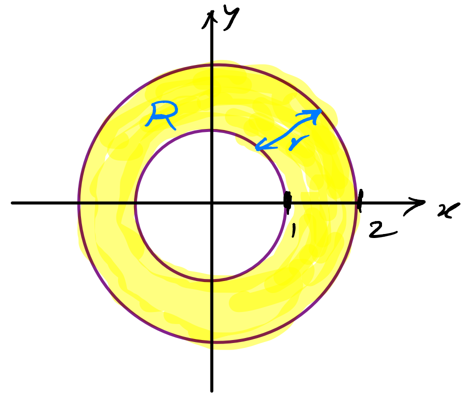
$$= \left(\int_0^{\frac{\pi}{2}} d\theta\right)\left(\int_0^2 r\sqrt{1+r^2}\, dr\right)$$

$u = 1+r^2 \Rightarrow du = 2r\,dr$

$$= \frac{\pi}{2} \cdot \frac{1}{2} \int \sqrt{u}\, du = \frac{\pi}{4} \cdot \frac{2}{3} u^{\frac{3}{2}}\Big| = \frac{\pi}{6}(1+r^2)^{\frac{3}{2}}\Big|_0^2$$

$$= \frac{\pi}{6}\left[5^{\frac{3}{2}} - 1\right]$$

Example 8 (15.3)**.** *Evaluate the integral* $\iint_R \frac{x^2}{x^2+y^2}\,dA$, *where* R *is the region between the circles* $x^2+y^2=1$ *and* $x^2+y^2=4$.



$R = \{(r,\theta) : 1 \le r \le 2,\ 0 \le \theta \le 2\pi\}$

$$\iint_R \frac{x^2}{x^2+y^2}\,dA = \int_0^{2\pi}\int_1^2 \frac{(r\cos\theta)^2}{r^2}\cdot r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_1^2 r\cos^2\theta\,dr\,d\theta$$

$$= \left(\int_0^{2\pi}\cos^2\theta\,d\theta\right)\left(\int_1^2 r\,dr\right)$$

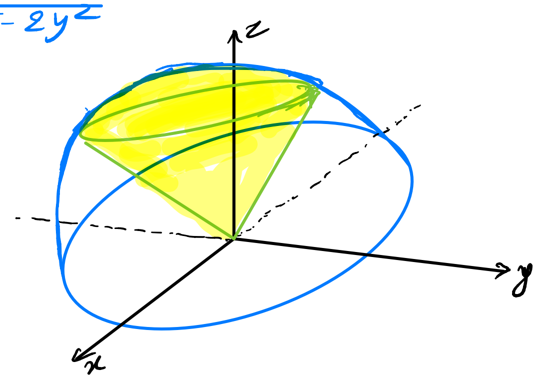
$\boxed{\cos 2\theta = 2\cos^2\theta - 1}$

$\Rightarrow \cos^2\theta = \frac{1+\cos 2\theta}{2}$

$$= \left(\int_0^{2\pi}\left(\frac{1}{2}+\frac{1}{2}\cos 2\theta\right)d\theta\right)\cdot \left.\frac{r^2}{2}\right|_{r=1}^{2}$$

$$= \frac{1}{\cancel{2}}\cdot\cancel{2}\pi\cdot\left[2-\frac{1}{2}\right] = \frac{3\pi}{2}.$$

Example 9 (15.3). *Find the volume of the solid between the cone* $z = \sqrt{x^2+y^2}$ *and the ellipsoid* $2x^2+2y^2+z^2=12$. $\Rightarrow$ $z = \sqrt{12-2x^2-2y^2}$



The intersection of $z=\sqrt{x^2+y^2}$ and

$2x^2+2y^2+z^2=12$ is

$2x^2+2y^2+x^2+y^2=12$

$\boxed{x^2+y^2=4}$

So, the projection of the solid onto xy-plane is a circular disk $x^2+y^2 \le 4$.

That is $D = \{(r,\theta) \mid 0 \le r \le 2,\ 0 \le \theta \le 2\pi\}$.

$$\text{Volume} = \iint_D (\text{top} - \text{bottom})\, dA = \iint_D \left[\sqrt{12-2x^2-2y^2} - \sqrt{x^2+y^2}\right] dA$$

$$= \int_0^{2\pi}\int_0^2 \left[\sqrt{12-2r^2} - r\right] r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^2 \sqrt{12-2r^2}\, r\,dr\,d\theta - \int_0^{2\pi}\int_0^2 r^2\,dr\,d\theta$$

$u = 12-2r^2 \Rightarrow du = -4r\,dr$

$$= (2\pi)\cdot\frac{-1}{4}\int \sqrt{u}\,du \quad - \quad 2\pi\left[\frac{r^3}{3}\right]_0^2$$

$$= -\frac{\pi}{2}\cdot\frac{2}{3}(12-2r^2)^{3/2}\Big|_{r=0}^{2} - \frac{16\pi}{3}$$

$$= \frac{-\pi}{3}\left[8-(12)^{3/2}\right] - \frac{16\pi}{3}$$

Example 10 (15.3). *Evaluate the integral* $\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\, dy\, dx$

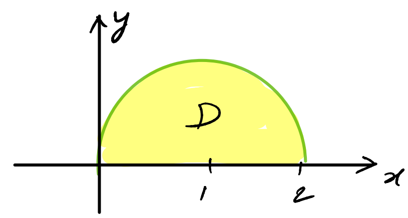
$D = \{(x,y) : 0 \le x \le 2,\ 0 \le y \le \sqrt{2x-x^2}\}$

$y = \sqrt{2x-x^2} \Rightarrow \boxed{x^2+y^2 = 2x}$

Completing the square, $x^2 - 2x + 1 + y^2 = 1$

$(x-1)^2 + (y-0)^2 = 1,$

which is a circle centered at $(1,0)$ with radius 1.



$x^2+y^2 = 2x \Rightarrow r^2 = 2r\cos\theta \Rightarrow r = 0$ or $r = 2\cos\theta$

Clearly, $0 \le \theta \le \frac{\pi}{2}$ $(x \ge 0, y \ge 0)$.

So, $D = \{(r,\theta) : 0 \le r \le 2\cos\theta,\ 0 \le \theta \le \frac{\pi}{2}\}$.

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\, dy\, dx = \int_0^{\frac{\pi}{2}} \int_0^{2\cos\theta} \sqrt{r^2} \cdot r\, dr\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{r^3}{3}\Big|_{r=0}^{2\cos\theta} d\theta = \frac{8}{3}\int_0^{\frac{\pi}{2}} \cos^3\theta\, d\theta$$

$$= \frac{8}{3}\int_0^{\frac{\pi}{2}} \cos\theta(1-\sin^2\theta)\, d\theta = \frac{8}{3}\int_0^{\frac{\pi}{2}} \cos\theta - \cos\theta\sin^2\theta\, d\theta.$$

$u = \sin\theta$
$du = \cos\theta\, d\theta$

$$= \frac{8}{3}\left[\int_0^{\frac{\pi}{2}} \cos\theta\, d\theta - \int u^2\, du\right]$$

$$= \frac{8}{3}\left[1 - \frac{1}{3}\sin^3\theta\Big|_{\theta=0}^{\frac{\pi}{2}}\right] = \frac{8}{3}\left(\frac{2}{3}\right) = \frac{16}{9}$$